Design of Optimal Orthogonal Tree-Structured Filter Banks

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Abstract— **A number of techniques have been proposed for the design of tree-structured filter banks matched to the statistics of the input signal. Amongst all of the proposed techniques, a top-down design approach has been recognized as a computationally efficient method for the design of tree-structured filter banks. However, the conventional top-down technique is sub-optimal. In this paper, we modify the existing top-down design approach to deal with its sub-optimal nature. Under certain assumptions, analytical expressions are derived for filter banks of unconstrained lengths that maximize the overall coding gain of the tree-structured filter bank.**

I. Introduction

Uniform and non-uniform tree-structured filter banks are used in a number of image and audio compression algorithms. The extensive use of tree-structured filter banks in source coding algorithms provides the motivation for developing design procedures for such filter banks. In a tree-structured filter bank, the input signal is decomposed into M subbands. Each or some of the M subbands are recursively decomposed further till the depth of the tree, reaches a certain pre-defined limit. The number of bits available to the input signal are distributed amongst the various subbands. The problem of designing tree-structured filter banks involves determining which subbands need to be split, the number of bits assigned to the subbands and coefficients of the filter banks used to carry out the M-band decomposition at every subband which is decomposed further. The objective of the design procedure is to chose these parameters such that the overall coding gain of the tree-structured filter bank is maximized, given the power spectral density (psd) of the input signal. For the rest of the paper, we use the term "node" to refer to a particular subband and "splitting a node" to refer to an M-band decompostion of the subband. It is assumed that an M-channel paraunitary filter bank, with possibly different coefficients, is used at all the nodes, in the tree which are split further.

Two possible approaches can be used for the design of tree structured filter banks. One of the approaches, referred to as the "bottom-top" approach, involves starting the design procedure at the leaf nodes (nodes that are not split any further) and then designing the root nodes. However, a problem with the bottom-top approach is the fact that we need to know the psd of the input at the leaf nodes. But the psd of the input to the leaf nodes can be determined only when the filter banks, at nodes above the leaf nodes have been designed. This poses a potential problem in the bottom-top approach since it would seem that we need to design the filter banks at nodes above the leaf nodes before filter banks at the leaf nodes are determined. Iterative techniques can be used to circumvent the problem, however such techniques tend to be computationally complex. In [1] a bottom-top algorithm is used to determine the optimal non-uniform decomposition (called wavelet packet) under the assumption that a fixed filter bank is used at every node of the decomposition.

The second technique, referred to as the "top-down" approach, involves designing filter banks at the root nodes before designing those at the leaf nodes. In some of the proposed top-down approaches [3], [4], each node is visited in a top-down fashion, starting from the root node. The coefficients of the filters at a node are obtained by maximizing the coding gain provided by the filter bank at the node. If the coding gain provided by the filter bank is greater than one, then the node is split further. The total number of bits allocated to the node are distributed amongst the M "child" nodes. This procedure is recursively carried out at all the nodes till all the leaf nodes have the property that the coding gain provided by the optimal filter bank at these nodes is close to unity. A similar top-down approach is also used in [2] to design the filter bank coefficients of a fixed non-uniform tree-structured decomposition. The top-down approach is computationally simpler than the iterative bottom-top approach, but suffers from the disadvantage of being sub-optimal.

The sub-optimal nature of the top-down approach in [2], [3], [4] (hereafter referred to as the conventional top-down approach) can be illustrated by means of an exam-

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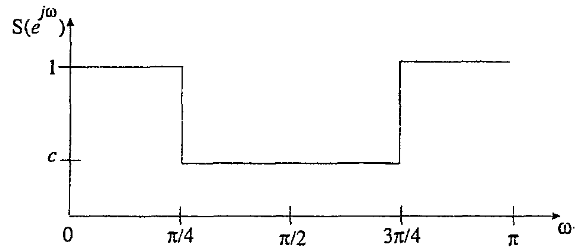

Fig. 1. Power Spectral density of the input to one of the nodes in the tree

ple. Consider the case when $M = 2$ and the input psd to one of the nodes in the tree, is as shown in Figure 1. The maximum coding gain provided by any two-channel orthogonal filter bank, (even the ideal brick-wall orthogonal filter bank) for the given input psd, is equal to one. Thus in the conventional top-down approach this node will not be split any further. However, the decision is suboptimal because if we split the node in question and also split each of the child nodes obtained, then a coding gain of $0.5(\sqrt{c} + \frac{1}{\sqrt{c}})$ can be achieved by the ideal brick-wall filter bank. This simple example illustrates the fact that the conventional top-down approach is "greedy" since it does not consider the possibility that even though splitting a node does not provide coding gain greater than one, higher coding gains might be achieved by splitting the node and its child nodes. The reason for this greedy behaviour is the fact that the coding gain provided by the tree at the child nodes is not included in the design of the parent node.

In this paper,we develop an algorithm for the design of orthogonal tree-structured filter banks which takes into account an estimate of the coding gain provided by the child nodes when the filter bank for a particular node is designed. This allows the proposed algorithm to avoid the sub-optimal nature of the conventional top-down design approach.

II. Proposed Design Procedure

We use a top-down approach to design orthogonal tree-structured filter bank. In the proposed algorithm, we make use of the "spectral flatness" [5] measure to decide whether to split a node or not. This is unlike the conventional top-down approach which uses the coding gain provided by the M-channel filter bank to decide whether to split a node or not. In the proposed design algorithm, we compute the value of spectral flatness measure, given by

$$\gamma_x = exp(\int_{-\pi}^{\pi} log_e S_{xx}(e^{j\omega})\ \frac{d\omega}{2\pi})/\sigma_x^2, \qquad (1)$$

at every node. If the value of $1/\gamma_x$ is greater than one, then the node is split. The rationale behind the use of spectral flatness measure is that if $1/\gamma_x$ is equal to one, then the spectrum is flat and hence there is no gain in splitting the node any further. However, if the value of $1/\gamma_x$ is greater than one, then we can potentially obtain higher coding gains by splitting the node even if the immediate split into M-subbands provides a coding gain of only one (it is possible that splitting one of the child nodes obtained by the split of the node could provide a large coding gain). Using the inverse of spectral flatness measure to decide whether to split a node or not, is equivalent to using the coding gain provided by an infinite channel filter bank at the node. If the coding gain provided by the infinte channel filter bank is not greater than one, then there is no gain in splitting the node any further. We compute the inverse of spectral flatness measure at every node in the tree. If this turns out to be greater than $1+\epsilon$, where ϵ is a pre-determined quantity, then the node is split further.

Once the decision to split a node has been made, the next step is to find the bit allocation amongst the M child nodes and the coefficients of the M-channel filter bank used to split the node into M subbands (nodes). Consider the case when we have made the decision to split a node n_i and let b_i be the number of bits that were allocated to the node n_i. We need to determine the bit allocation b_{ik} $k = 0, 1, \ldots, M-1$ amongst the M child nodes and the filter bank used to split the node. The constraint on the bit allocation is

$$\frac{1}{M}\sum_{k=1}^{M} b_{ik} = b_i. \qquad (2)$$

If the filter bank used to split the node n_i is orthonormal then the variance of the signal at n_i is related to the variance of the signals at its child nodes, n_{ik} through

$$\sigma_{x_i}^2 = \sum_{k=1}^{M} \sigma_{x_{ik}}^2. \qquad (3)$$

This is due to the fact that the analysis filters in an M-channel orthonormal filter bank satisfy the property

$$\sum_{k=1}^{M} |H_{ik}(e^{j\omega})|^2 = 1, \qquad (4)$$

where, $H_{ik}(e^{j\omega})$ represents the frequency response of the kth analysis filter. If the child nodes of n_i were not being split, then the distortion due to the quantization of signals at the child nodes would be

$$D_i = \sum_{k=1}^{M} c\ \sigma_{x_{ik}}^2 2^{-2b_{ik}}, \qquad (5)$$

where b_{ik} is the number of bits allocated to the kth child node, $\sigma^2_{x_{ik}}$ is the variance of the signal at the kth child node and c is a constant. The presense of a possible tree structure at one or more child nodes, say n_{im}, causes the variance of the quantization noise at n_{im} to be reduced by a factor equal to the coding gain provided by the tree structure at n_{im}. Thus the value of distortion due to quantization of subband signals at node n_i is

$$D_i = \sum_{k=1}^{M} \frac{c\, \sigma^2_{x_{ik}} 2^{-2b_{ik}}}{G_{ik}},$$

where G_{im} is the coding gain provided by the tree at node n_{im}. The coding gain provided by the tree structure at n_{im} cannot be determined untill the filter bank and bit allocation at the parent node n_i is determined. However, we can estimate the coding gain at each child node by the value of the asymptotic coding gain, G^{∞}_{im}, at that node. The justification for using G^{∞}_{im} as an estimate of the coding gain provided by the tree at node n_{im} is the fact that the tree starting from this node will be grown till the coding gain provided by the tree is closed to its asymptotic value. The estimated distortion at the node n_i can be written as

$$\begin{aligned} D_i &= \sum_{k=1}^{M} \frac{c\, \sigma^2_{x_{ik}} 2^{-2b_{ik}}}{G^{\infty}_{ik}}, \\ &= \sum_{k=1}^{M} c\ exp(\int_{-\pi}^{\pi} log_e(S_{x_{ik}x_{ik}}(e^{j\omega}))\frac{d\omega}{2\pi})\ 2^{-2b_{ik}} \end{aligned} \quad (6)$$

The use of G^{∞}_{im} as an estimate of G_{im} is another difference between the proposed approach and the conventional top-down approach. In the conventional top-down approach G_{im} is set to one when the filter bank at n_i is designed. The bit allocation amongst the child nodes of n_i is done such that D_i is minimized subject to the bit constraint given in Eq. (2). The optimal bit allocation is given by

$$b_{ik} = b_i + \frac{1}{2} log_2 \left(\frac{\gamma_{x_{ik}} \sigma^2_{x_{ik}}}{(\prod_{j=1}^{M} \gamma_{x_{ij}} \sigma^2_{x_{ij}})^{1/M}} \right), \quad (7)$$

where, $\gamma_{x_{ik}} = exp(\int_{-\pi}^{\pi} (log_e S_{x_{ik}x_{ik}}(e^{j\omega}))\ \frac{d\omega}{2\pi})/\sigma^2_{x_{ik}}$. The value of the distortion D_i under optimal bit allocation, as given by Eq. (7), is

$$D_i = Mc \left(\prod_{k=1}^{M} \gamma_{x_{ik}} \sigma^2_{x_{ik}} \right)^{1/M} 2^{-2b_i}. \quad (8)$$

The coefficients of the filter bank at n_i are designed such that the value of D_i given in Eq. (8) is minimized subject to the constraint that the resulting filter bank is orthonormal. This can be done by using standard algorithms for the design of M-channel paraunitary filter banks [6]. Once the coefficients of the filter bank at n_i have been optimized, it is possible to determine the psd of the signals at the M child nodes. The number of bits allocated to each child node can then be determined using the bit-allocation result given in Eq. (7). A similar procedure can be recursively carried out at every node in the tree, till all the leaf nodes have the property that the value of the inverse of spectral flatness measure at each of these nodes is equal (close) to unity.

III. Characterization of Optimal Filter Banks

In this section we determine the optimal filter bank at n_i that minimizes Eq. (8), under the assumption that there is no constraint on the length of the filters. The expression for D_i can be simplified as

$$\begin{aligned} D_i &= cM\ exp(\sum_{k=1}^{M} \int_{-\pi}^{\pi} log_e(S_{x_{ik}x_{ik}}(e^{j\omega}))\frac{d\omega}{2\pi})^{1/M} 2^{-2b_i}, \\ &= cM\ exp(\frac{1}{M}\int_{-\pi}^{\pi} log_e \left(\prod_{k=1}^{M} S_{x_{ik}x_{ik}}(e^{j\omega}) \right) \frac{d\omega}{2\pi}) 2^{-2b_i} \\ &= cM\ exp(\frac{1}{M}\int_{-\pi}^{\pi} log_e \left(\prod_{k=1}^{M} (\mathbf{B}(e^{j\omega}))_{kk} \right) \frac{d\omega}{2\pi}) 2^{-2b_i}, \end{aligned}$$

where $\mathbf{B}(e^{j\omega}) = \mathbf{E}(e^{j\omega})\mathbf{S}_{x_ix_i}(e^{j\omega})\mathbf{E}^*(e^{j\omega})$, $(\mathbf{A})_{kk}$ denotes the diagonal element of the matrix $\mathbf{A}$, $\mathbf{E}(e^{j\omega})$ denotes the polyphase matrix of the analysis filter bank, $*$ is used to indicate the transpose conjugate of a matrix and $\mathbf{S}_{x_ix_i}(e^{j\omega})$ denotes the psd matrix [7], [8] of the signal at the input to the analysis poplyphase matrix at n_i. Minimizing D_i in the above equation is equivalent to finding the polyphase matrix transfer function $\mathbf{E}(e^{j\omega})$ such that

$$C_i = \int_{-\pi}^{\pi} log_e(\prod_{k=1}^{M} (\mathbf{E}(e^{j\omega})\mathbf{S}_{x_ix_i}(e^{j\omega})\mathbf{E}^*(e^{j\omega}))_{kk})\frac{d\omega}{2\pi}. \quad (9)$$

is minimized (since exponential is a monotonocally increasing function). Now

$$\begin{aligned} C_i &= \int_{-\pi}^{\pi} log_e(\prod_{k=1}^{M} (\mathbf{E}(e^{j\omega})\mathbf{S}_{x_ix_i}(e^{j\omega})\mathbf{E}^*(e^{j\omega}))_{kk})\frac{d\omega}{2\pi}, \\ &\geq \int_{-\pi}^{\pi} log_e(det\,(\mathbf{E}(e^{j\omega})\mathbf{S}_{x_ix_i}(e^{j\omega})\mathbf{E}^*(e^{j\omega})))\frac{d\omega}{2\pi}, \\ &= \int_{-\pi}^{\pi} log_e(det(\mathbf{S}_{x_ix_i}(e^{j\omega})))\frac{d\omega}{2\pi}, \end{aligned} \quad (10)$$

where, we have made use of the inequality, $\prod_{i=1}^{M} \mathbf{A}_{ii} \geq det(\mathbf{A})$, for a positive definite matrix with equality only if $\mathbf{A}$ is a diagonal matrix. The positive definiteness of $\mathbf{E}(e^{j\omega})\mathbf{S}_{x_ix_i}(e^{j\omega})\mathbf{E}^*(e^{j\omega})$ follows from the fact that $\mathbf{S}_{x_ix_i}(e^{j\omega})$ is a spectral matrix [8] and is hence positive definite. Thus the quantity C_i in Eq. (9) is minimized when the inequality becomes an equality. This happens when $\mathbf{E}(e^{j\omega_0})$ is chosen to be the eigenvector of the psd matrix $\mathbf{S}_{x_i x_i}(e^{j\omega_0})$ for each value of $\omega_0 \in [0, 2\pi)$. Thus the optimal filter bank at n_i that minimizes the distortion, D_i in Eq. (8), is given by the eigenvalue decomposition of the psd matrix of the input at n_i.

IV. Preliminary Results

The proposed design algorithm was used to determine the optimal tree structured filter bank for an AR(1) source with $\rho = 0.95$. The value of M was chosen to be equal to two. The optimal decomposition and the coefficients of the filter bank were determined, starting from the root node, as explained in Section 2. Figure 2 shows the value of the overall coding gain achieved versus the length of the filters. The value of ϵ was chosen to be 0.1. The figure shows the overall coding gain of the tree structure for different values of the depth upto which the tree structures are grown. The depth of the tree refers to the depth of the last leaf node starting from the root node (which is assigned a depth of one). The maximum value of coding gain that can be obtained for the AR(1) process with $\rho = 0.95$ is 10.25. The results in Figure 2 indicate that for filters of a given length, the value of the coding gain obtained from the tree increases with the depth of the tree. But when the depth of the tree increases beyond a certain limit, the coding gain of the tree saturates at a value that depends upon the length of the filters used.

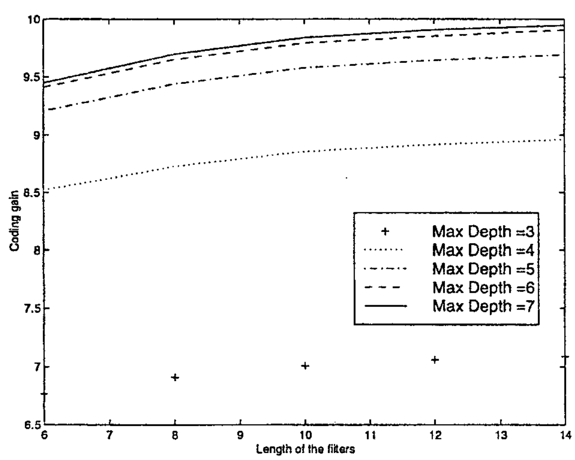

Fig. 2. The value of coding gain versus the length of the filters for different depths of the tree

V. Concluding Remarks

In this paper, we developed an algorithm for the design of tree structured filter banks. The use of the spectral flatness measure at each node, rather than the coding gain of an M-channel filter bank is proposed as a criterion to determine the decomposition. The coefficients of the filter banks are computed at every node by minimizing a cost function that takes into account an estimate of the coding gains provided by the tree at the child nodes. When there are no constraints on the length of the filter bank, it is shown that the optimal orthogonal filter bank at each node is determined by the eigenvector decomposition of the psd matrix at that node.

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