A NEW APPROACH TO THE DESIGN OF TWO-CHANNEL PERFECT RECONSTRUCTION FILTER BANKS*

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ABSTRACT

In this paper, we propose a new technique for the design of two-channel perfect reconstruction filter banks (PRFBs). The proposed design approach is generic, and can be applied to the design of both orthogonal as well as linear phase biorthogonal PRFBs. Unlike previous design techniques where the perfect reconstruction (PR) property was structurally imposed, our design starts with trivial filters that yield perfect reconstruction. The length of the filters is subsequently increased to improve their magnitude reponse without sacrificing either the linear phase/orthogonality or the perfect reconstruction property.

1. INTRODUCTION

Subband coding has been used for the compression of speech, audio and images. The input signal is split into different subbands using a set of bandpass filters. These subband signals are then downsampled and quantized. The quantized coefficients are transmitted to the decoder, where they are upsampled and combined appropriately using the synthesis filters. In the absence of quantization of the subband components, the filter bank introduces two kinds of errors – aliasing distortion and magnitude or phase distortion. By choosing the analysis and synthesis filters appropriately these errors can be eliminated resulting in a perfect reconstruction filter bank (PRFB). In addition, certain applications also require that the individual filters in a PRFB have linear phase.

One approach to the design of two-channel PRFBs is based on spectral factorization of half-band filters [1]. Another approach involves the lattice factorization of paraunitary matrices [2], where the perfect reconstruction (PR) property was imposed structurally on the filter bank. Lattice-type structures [3] have been proposed to structurally enforce both the PR and the linear phase properties simultaneously. In all of these methods, since perfect reconstruction is ensured based on the structure of the filter bank, different design techniques may result for structurally different types (orthogonal, biorthogonal) of filter banks. There is also no clear way of transforming the design procedure for one type of PRFB to that for another.

In this paper, the PR property is not tied to the structure of the filter banks and thus, any PRFB can be designed using the same procedure. Our design starts with trivial filters that satisfy the PR property. The lengths of the filters are gradually increased so that filters with improved frequency responses are obtained while retaining the PR property.

*This work was supported by a University of California MICRO grant with matching support from Envision Medical Corporartion and Xerox Corporation.

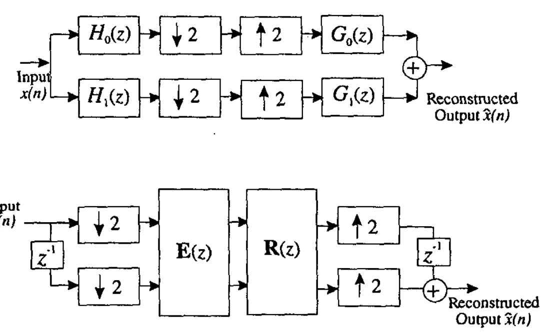

Figure 1: A two-channel filter bank and its equivalent polyphase decomposition

2. TWO-CHANNEL PERFECT RECONSTRUCTION FILTER BANKS

For the typical two-channel filter bank shown in Figure 1, the conditions to be satisfied by the analysis and the synthesis filters to eliminate aliasing and to achieve perfect reconstruction are

$$H_0(-z)G_0(z) + H_1(-z)G_1(z) = 0 \qquad (1)$$

and

$$H_0(z)G_0(z) + H_1(z)G_1(z) = \alpha z^{-r}. \qquad (2)$$

By applying a polyphase decomposition to the filters, a sufficient condition to achieve PR is given by [4]

$$\mathbf{R}(z)\mathbf{E}(z) = \beta z^{-m_o}\mathbf{I}. \quad (3)$$

A necessary and sufficient condition [4], to satisfy Eq. (3), using FIR filters is

$$\det \mathbf{E}(z) = \alpha z^{-K}, \quad (4)$$

where α is a non-zero constant and K is an integer. If an $\mathbf{E}(z)$ that satisfying Eq. (4) can be found, then the corresponding $\mathbf{R}(z)$ is given by $\mathbf{R}(z) = \beta z^{-m_o}\mathbf{E}^{-1}(z)$. There exist a number of solutions to Eq. (4), and it is not difficult to find an $\mathbf{E}(z)$ that satisfies Eq. (4). The difficulty in the design of a PRFB lies in finding a $\mathbf{E}(z)$ that satisfies Eq. (4) and minimizes a certain cost function (typically based on the magnitude response of the filters).

We pose the problem of the design of PRFB as (i) choosing an initial solution to Eq. (4), and (ii) subsequently designing an efficient method to proceed from one solution to others, thereby eventually arriving at the optimal solution. We also show that for the special cases of orthogonal and linear phase biorthogonal filter banks, all possible solutions to Eq. (4) can be generated using this approach. For the initial solution, we choose filters whose polyphase matrix $\mathbf{E}(z)$ is of the form

$$\mathbf{E}(z) = \begin{bmatrix} k_0 & k_1 \\ k_2 & k_3 \end{bmatrix}. \quad (5)$$

The initial filters are of length 2, and thus do not have good magnitude responses. To obtain filters with good magnitude responses, we need to increase the lengths of $H_0(z)$ and $H_1(z)$, or equivalently, those of $E_{00}(z)$, $E_{01}(z)$, $E_{10}(z)$ and $E_{11}(z)$, where $E_{ij}(z)$ represents the $(i+1, j+1)$ element of $\mathbf{E}(z)$. In the first step, the lengths of $E_{00}(z)$ and $E_{01}(z)$ are increased by adding the polynomials $\alpha(z)$ and $\beta(z)$ respectively. The lengths of $\alpha(z)$ and $\beta(z)$ could be greater than those of $E_{00}(z)$ and $E_{01}(z)$, respectively. The polynomials $\alpha(z)$ and $\beta(z)$ should be chosen such that the new polyphase components satisfy Eq. (4), thereby leading to

$$\alpha(z) = P(z)E_{10}(z) \quad \text{and} \quad \beta(z) = P(z)E_{11}(z). \quad (6)$$

Hence the new polyphase components $E'_{00}(z)$ and $E'_{01}(z)$ are given by

$$\begin{aligned} E'_{00}(z) &= E_{00}(z) + P(z)E_{10}(z) \\ E'_{01}(z) &= E_{01}(z) + P(z)E_{11}(z). \end{aligned} \quad (7)$$

In the next step, the length of $H_1(z)$ is increased similarly, resulting in

$$\begin{aligned} E'_{10}(z) &= E_{10}(z) + Q(z)E'_{00}(z) \\ &= E_{10}(z)(1 + P(z)Q(z)) + Q(z)E_{00}(z), \end{aligned} \quad (8)$$

and

$$\begin{aligned} E'_{11}(z) &= E_{11}(z) + Q(z)E'_{01}(z) \\ &= E_{11}(z)(1 + P(z)Q(z)) + Q(z)E_{01}(z). \end{aligned} \quad (9)$$

This method of constructing $H'_0(z)$ and $H'_1(z)$ ensures perfect reconstruction regardless of the actual values of $P(z)$, $Q(z)$, k_0, k_1, k_2 and k_3. The design problem for two-channel PRFBs can be reformulated as: choosing $P(z)$, $Q(z)$, k_0, k_1, k_2 and k_3 to minimize the cost function given by

$$\phi = \int_0^{\omega_p} (1 - | H'_0(e^{j\omega}) |)^2 d\omega + \int_{\omega_s}^{\pi} | H'_0(e^{j\omega}) |^2 \, d\omega + \int_{\omega_p}^{\pi} (1 - | H'_1(e^{j\omega}) |)^2 d\omega + \int_0^{\omega_s} | H'_1(e^{j\omega}) |^2 \, d\omega \quad (10)$$

Note that k_0, k_1, k_2 and k_3 should be chosen such that $k_0k_3 \neq k_1k_2$. If such a situation arises one could modify any of the k_i's by adding a small constant δ. Different types of PRFBs may place different constraints on $P(z)$, $Q(z)$, k_0, k_1, k_2 and k_3. However, the procedure outlined above is generic and is thus applicable to the design of two-channel PRFBs of any type. In this paper, we apply this procedure to the design of orthogonal PRFBs and linear phase PRFBs.

3. LINEAR PHASE PERFECT RECONSTRUCTION FILTER BANKS

First, consider the case of analysis filters having equal lengths. It can be shown that in this case the lowpass and the highpass analysis filters are of even length and while the lowpass filter is symmetric, the highpass filter is anti-symmetric [3]. The various polyphase components are related by

$$E_{01}(z) = z^{-L+1}E_{00}(z^{-1}), \quad (11)$$

and

$$E_{11}(z) = -z^{-L+1}E_{10}(z^{-1}), \quad (12)$$

where the length of the filters $H_0(z)$ and $H_1(z)$ is $2L$. To design the PRFB, we choose $H_0(z) = k_0(1 + z^{-1})$ and $H_1(z) = k_1(1 - z^{-1})$ as the initial solution. The polyphase matrix is then given by

$$\mathbf{E}(z) = \begin{bmatrix} k_0 & k_0 \\ k_1 & -k_1 \end{bmatrix}. \quad (13)$$

Assume now that we have designed a linear phase PRFB of length $2l$. The lengths of $H_0(z)$ and $H_1(z)$ are now increased by 2, while ensuring that Eq. (4) remains valid. This is done by adding polynomials $\alpha(z)$ and $\beta(z)$, respectively, of length $2l+2$, resulting in the new filters $H'_0(z)$ and $H'_1(z)$. For $H'_0(z)$ to be symmetric, and $H'_1(z)$ to be antisymmetric, the conditions to be satisfied are as follows:

- $\alpha(z)$ is symmetric and $\beta(z)$ is anti-symmetric,
- The initial solutions $H_0(z)$ and $H_1(z)$ are "centered", i.e., $H_0(z)$ and $H_1(z)$ are multiplied by z^{-1}, resulting in

$$H_0'(z) = z^{-1}H_0(z) + \alpha(z),$$
$$H_1'(z) = z^{-1}H_1(z) + \beta(z).$$

In terms of the polyphase components of $H_0'(z)$ and $H_1'(z)$, we obtain

$$\begin{aligned} E_{00}'(z) &= z^{-1}E_{01}(z) + \alpha_0(z), \\ E_{01}'(z) &= E_{00}(z) + \alpha_1(z), \\ E_{10}'(z) &= z^{-1}E_{11}(z) + \beta_0(z), \\ E_{11}'(z) &= E_{10}(z) + \beta_1(z), \end{aligned} \tag{14}$$

where $\alpha(z) = \alpha_0(z^2) + z^{-1}\alpha_1(z^2)$ and $\beta(z) = \beta_0(z^2) + z^{-1}\beta_1(z^2)$. The new value of det $\mathbf{E}'(z)$ is given by

$$\begin{aligned} &\det \mathbf{E}'(z) = -z^{-1}\det \mathbf{E}(z) + \alpha_0(z)E_{10}(z) \\ &+z^{-1}E_{01}(z)\beta_1(z) + \alpha_0(z)\beta_1(z) - \beta_0(z)E_{00}(z) \\ &-z^{-1}E_{11}(z)\alpha_1(z) - \alpha_1(z)\beta_0(z). \end{aligned} \tag{15}$$

Due to the symmetry of $\alpha(z)$ and the anti-symmetry of $\beta(z)$, we require that $\alpha_1(z) = z^{-l}\alpha_0(z^{-1})$ and $\beta_1(z) = -z^{-l}\beta_0(z^{-1})$. If $\alpha_0(z) = k_l E_{00}(z)$ and $\beta_0(z) = k_l E_{10}(z)$, then det $\mathbf{E}'(z) = (k_l^2 - 1)z^{-1}\det \mathbf{E}(z)$. This implies that the perfect reconstruction condition is satisfied but for a scaling factor and an additional delay. The lengths of $H_0(z)$ and $H_1(z)$ can thus be recursively increased in this way, starting with filters of length 2, while maintaining the PR property. The parameters $\{k_l\}$'s can be chosen to minimize the cost function given in Eq. (10). The resulting structure of the polyphase matrix is similar to that reported in [3], where it was been shown that the structure generated "almost" every PRFB where the analysis filters are equal length, linear phase filters. Furthermore, the lattice-type structures are specific to the design of linear phase PRFBs, while our method can be used to design any type of PRFB.

Next, consider the case of analysis filters having different but even lengths. In this case, it can be shown that the lengths of $H_1(z)$ and $H_0(z)$ differ by an even multiple of two [3]. Assume that the lengths of $H_1(z)$ and $H_0(z)$ are $2L$ and $2L + 4K$, respectively. By applying the above procedure to filters of length 2, $H_1(z)$ and $H_0(z)$ are constructed till their lengths equal $2L$, while ensuring that the PR property continues to be satisfied. Then, the length of $H_0(z)$ alone is increased by "centering" $H_0(z)$, and then adding a symmetric polynomial $\alpha(z)$ of length $2L + 4K$. The new polyphase components are now given by

$$\begin{aligned} E_{00}'(z) &= z^{-K}E_{00}(z) + \alpha_0(z), \\ E_{01}'(z) &= z^{-K}E_{01}(z) + \alpha_1(z), \\ E_{10}'(z) &= E_{10}(z), \end{aligned}$$

and

$$E_{11}'(z) = E_{11}(z), \tag{16}$$

where $\alpha(z) = \alpha_0(z^2) + z^{-1}\alpha_1(z^2)$ and

$$\alpha_1(z) = z^{-(L+2K-1)}\alpha_0(z^{-1}).$$

For perfect reconstruction, $\alpha_0(z)$ and $\alpha_1(z)$ should satisfy

$$\frac{\alpha_0(z)}{\alpha_1(z)} = \frac{E_{10}(z)}{E_{11}(z)}. \tag{17}$$

One solution is given by $\alpha_0(z) = P(z)E_{10}(z)$ and $\alpha_1(z) = P(z)E_{11}(z)$, where $P(z)$ is a polynomial of length $2K+1$. The requirement $E_{01}'(z) = z^{-(L+2K-1)}E_{00}'(z^{-1})$ enforces the relation $P(z) = -z^{-2K}P(z^{-1})$. Finally, $\{k_l\}$'s and $P(z)$ are jointly optimized to minimize the cost function given in Eq. (10).

4. ORTHOGONAL FILTER BANKS

The design procedure outlined in Section 2 can be modified for the design of orthogonal (or paraunitary) PRFB. For achieving perfect reconstruction, the conditions on the two polyphase components can be shown to be

$$E_0(z)E_0(z^{-1}) + E_1(z)E_1(z^{-1}) = \text{const} \tag{18}$$

The initial solution is chosen as

$$\mathbf{E}(z) = \begin{bmatrix} k_0 & k_1 \\ -k_1 & k_0 \end{bmatrix}. \tag{19}$$

To increase the lengths of filters, the polyphase components are updated as

$$\begin{aligned} E_0^{new}(z) &= z^{-r}E_0(z) + \alpha(z), \\ E_1^{new}(z) &= z^{-p}E_1(z) + \beta(z), \end{aligned} \tag{20}$$

where the polynomials $\alpha(z)$ and $\beta(z)$ are chosen such that (18) is satisfied by $E_0^{new}(z)$ and $E_1^{new}(z)$. If $\alpha(z)$ and $\beta(z)$ are chosen such that

$$\alpha(z) = kz^{-p}E_1(z) \text{ and } \beta(z) = -kz^{-r}E_0(z), \tag{21}$$

where k is a free parameter, then, $E_0^{new}(z)$ and $E_1^{new}(z)$ satisfy (18). In our design procedure, (20) and (21) are recursively applied to obtain filters of increasing length (starting from the initial solution in (19) of length 2). For every recursive step, the length of the filters increases by $\max(2r, 2p)$ and one additional free parameter becomes available.

To obtain filters with good magnitude response, we need as many free parameters as permitted by the structure of the filter bank. Thus, if $\max(r,p) = 1$ at every step of the recursion, the length of the filters increases by 2 at each step, and we obtain the maximum number of free parameters at the end of the recursion. In particular, if we choose $r = 0$ and $p = 1$, then,

$$\begin{bmatrix} E_0^{new}(z) \\ E_1^{new}(z) \end{bmatrix} = \begin{bmatrix} 1 & k \\ -k & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix} \begin{bmatrix} E_0(z) \\ E_1(z) \end{bmatrix},$$

which is equivalent to the lattice structure developed for the factorization of paraunitary matrices [2]. Furthermore, it has been established that all paraunitary matrices can be generated using this lattice structure. Since a necessary and sufficient condition for a filter bank to be orthogonal is that its polyphase matrix be paraunitary, it follows that all possible orthogonal filter banks can be generated by our approach.

5. RESULTS

The above procedure has been applied to the design of a PR filter bank where the analysis filters are of length 32 each. Two cases are illustrated here. In the first case, the analysis filters in the filter bank have linear phase while in the second case, the analysis filters are such that the overall filter bank is orthogonal.

Figure 2 shows the magnitude responses of the lowpass and the highpass analysis filters when the analysis filters are constrained to have linear phase. Figure 3 shows the magnitude response of the analysis filters of length 32 each in an orthogonal PRFB. These results indicate that the filters designed by our methods are equivalent to those designed using alternative techniques.

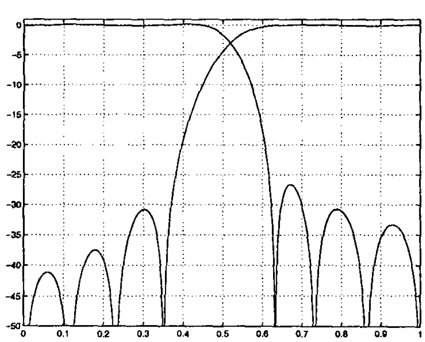

Figure 2: Magnitude response of linear phase analysis filters of length 32 in a perfect reconstruction filter bank.

6. CONCLUDING REMARKS

In this paper, we present a new approach to the design of two-channel PRFBs. The key notion introduced is a procedure for the design of any type of PRFB, starting with trivial filters satisfying the PR property. Results for different types (orthogonal and linear phase biorthogonal) PRFBs indicate that the proposed technique can be applied to design filters with good magnitude response.

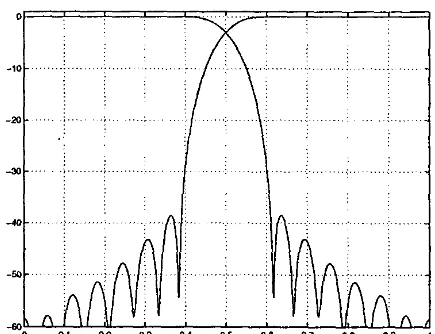

Figure 3: Magnitude response of the analysis filters of length 32 in an orthogonal perfect reconstruction filter bank.

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