AN AXIOMATIC APPROACH TO CORNER DETECTION

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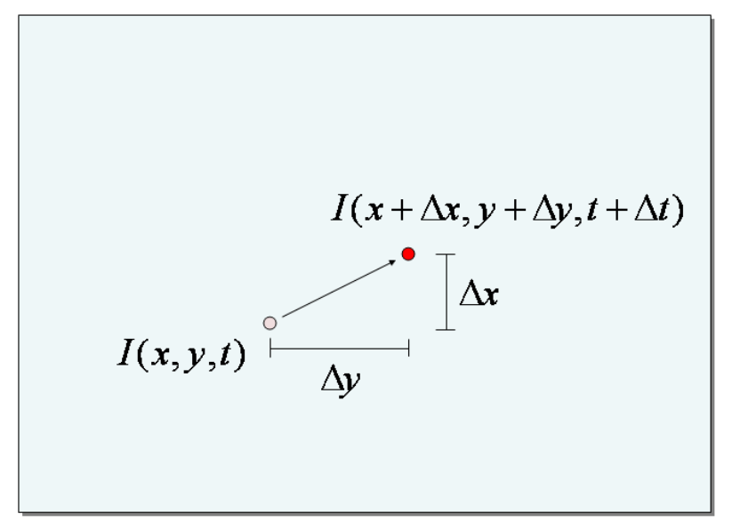
Goals

- Generalize corner detectors based on the gradient normal matrix
- Present a set of 4 axioms one might reasonably require a corner detector to satisfy
- Explore the restrictions and the consequences of the axioms on the generalized corner detectors
- See also [Kenney et al., 2003, Triggs, 2004, Zuliani et al., 2004]

The gradient normal matrix A^TA

Translational model:



- $\boldsymbol{v} \stackrel{\text{def}}{=} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}, \Delta t = 1$
- $I_x \stackrel{\text{def}}{=} \frac{\partial I}{\partial x}, I_y \stackrel{\text{def}}{=} \frac{\partial I}{\partial y}$
- $I_t \stackrel{\text{def}}{=} \frac{\partial I}{\partial t}, \boldsymbol{\eta} = \text{noise}$

From:

$$\underbrace{I(x,y,t) = I(x+\Delta x, y+\Delta y, t+\Delta t) + \boldsymbol{\eta}}_{\text{Brightness constraint equation}}$$

to:

$$\underbrace{\boxed{\begin{bmatrix} I_x^1 & I_y^1 \\ \vdots & \\ I_x^N & I_y^N \end{bmatrix}}}_{\boxed{A}} \boldsymbol{v} = - \begin{bmatrix} I_t^1 \\ \vdots \\ I_t^N \end{bmatrix} - \boldsymbol{\eta}$$

A simple case: no motion, just noise $\Rightarrow \boldsymbol{v}_{exact} = 0$ and $\boldsymbol{I}_t^i \equiv 0$. But in practice:

$$A\boldsymbol{v}_{\text{computed}} = -\boldsymbol{\eta}$$

The least square solution is given by:

$$\boldsymbol{v}_{\text{computed}} = -(A^TA)^{-1}A^T\boldsymbol{\eta}$$

Define the error $\boldsymbol{e} = \boldsymbol{v}_{\text{computed}} - \boldsymbol{v}_{\text{exact}}$

$$\boxed{\|\boldsymbol{e}\| \leq \|(A^TA)^{-1}A^T\| \, \|\boldsymbol{\eta}\|}$$

$\|(A^TA)^{-1}A^T\|$ controls the error multiplication factor. Key idea: choose points for which $\|(A^TA)^{-1}A^T\|$ is small. The importance of the gradient normal matrix A^TA is justified by:

$$\|(A^TA)^{-1}A^T\|_2^2 = \frac{1}{\lambda_{min}(A^TA)}$$

where:

$$\boxed{A^TA = \begin{bmatrix} \sum_{i=1}^N I_x^i I_x^i & \sum_{i=1}^N I_x^i I_y^i \\ \sum_{i=1}^N I_x^i I_y^i & \sum_{i=1}^N I_y^i I_y^i \end{bmatrix}}$$

Generalization

Consider a "generalized" image:

$$\begin{aligned} \boldsymbol{I} : \mathbb{R}^n &\to \mathbb{R}^m \\ \boldsymbol{x} &\mapsto \boldsymbol{I}(\boldsymbol{x}) \end{aligned}$$

- $n > 2 \to$ CT images
- $m > 1 \to$ multichannel images

The gradient normal matrix becomes:

$$\boxed{A^TA = \sum_{j=1}^N \sum_{i=1}^m \left(\nabla \boldsymbol{I}_i^j\right)^T \nabla \boldsymbol{I}_i^j \in \mathbb{R}^{n\times n}}$$

A corner detector is a function f of A^TA:

$$\begin{aligned} f : \mathcal{M}_n &\to \mathbb{R} \\ \boxed{A^TA} &\mapsto \boxed{f\left(\sum_{j=1}^N \sum_{i=1}^m \left(\nabla \boldsymbol{I}_i^j\right)^T \nabla \boldsymbol{I}_i^j\right)} \end{aligned}$$

where $\mathcal{M}_n$ is the space of the $n \times n$ positive semi-definite symmetric matrices.
Examples:

- Harris-Stephens [Harris and Stephens, 1988]:

$$f_{HS} \stackrel{\text{def}}{=} \det(A^TA) - \alpha\left(\text{trace}(A^TA)\right)^n = \prod_{i=1}^n \lambda_i - \alpha\left(\sum_{i=1}^n \lambda_i\right)^n$$

- Modified Rohr [Rohr, 1994, Rohr, 1997]:

$$f_R \stackrel{\text{def}}{=} \sqrt[n]{\det(A^TA)}$$

- Förstner [Förstner, 1986]:

$$f_F \stackrel{\text{def}}{=} \frac{1}{\text{trace}\left((A^TA)^{-1}\right) + \varepsilon} = \frac{1}{\varepsilon + \sum_{i=1}^n \frac{1}{\lambda_i}}$$

- Shi-Tomasi [Shi and Tomasi, 1994]:

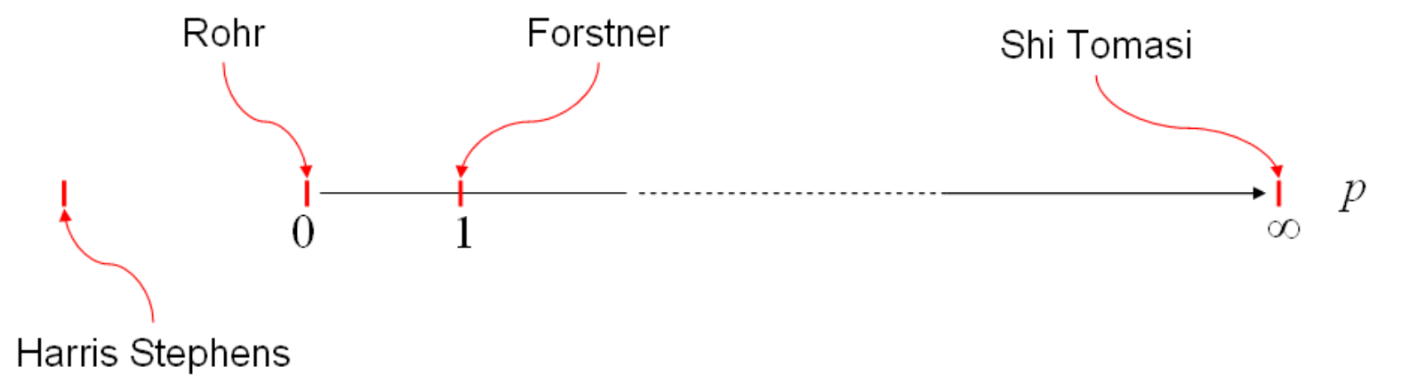
$$f_{ST} \stackrel{\text{def}}{=} \lambda_{min}(A^TA)$$

- Kenney et al. :

$$f_{Kenney,p} = \frac{1}{\left\|(A^TA)^{-1}\right\|_p} = \frac{1}{\left(\sum_{i=1}^n \frac{1}{\lambda_i^p}\right)^{\frac{1}{p}}}$$

Detectors Equivalence

Except Harris Stephens, the above detectors are equivalent modulo the choice of a suitable matrix norm:



A Couple of Definitions

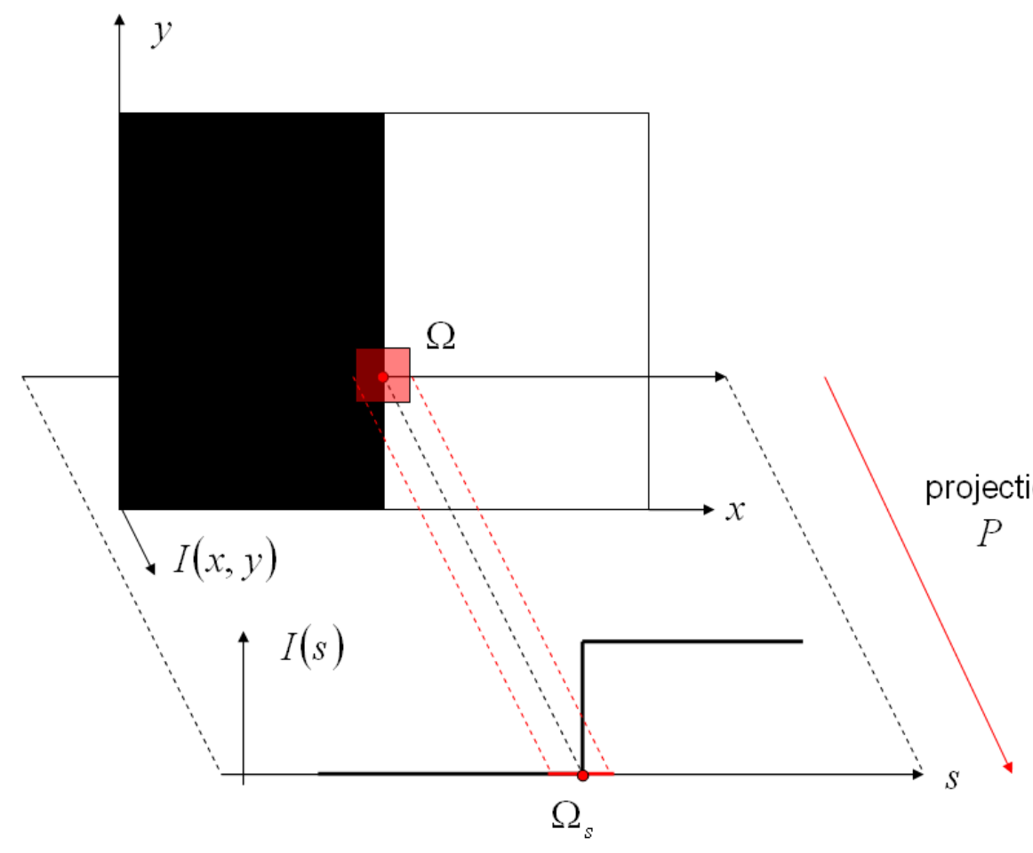
Definition 1 *Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A^TA at $\mathbf{x}$. We say that a set of points X in the image $\boldsymbol{I}$ has constant eigen-energy with respect to the q-norm if $\lambda_1^q + \cdots + \lambda_n^q$ is constant over $\mathbf{x} \in X$.*

Definition 2 *A point $\mathbf{x}$ is isotropic (with respect to the image $\boldsymbol{I}$) if the eigenvalues of A^TA are all equal: $\lambda_1 = \lambda_2 = \ldots = \lambda_n$.*

Axiom 1

Let $P \in \mathbb{R}^{n\times d_P}$ with $d_P \leq n$ be a matrix with orthonormal columns. Then $\boxed{f(A^TA) \leq f(P^TA^TAP)}$ and equality is achieved if and only if $\mathbf{x}$ is a point of isotropy (isotropy condition) or $d_P = n$ (rotation invariance condition).

Motivation:



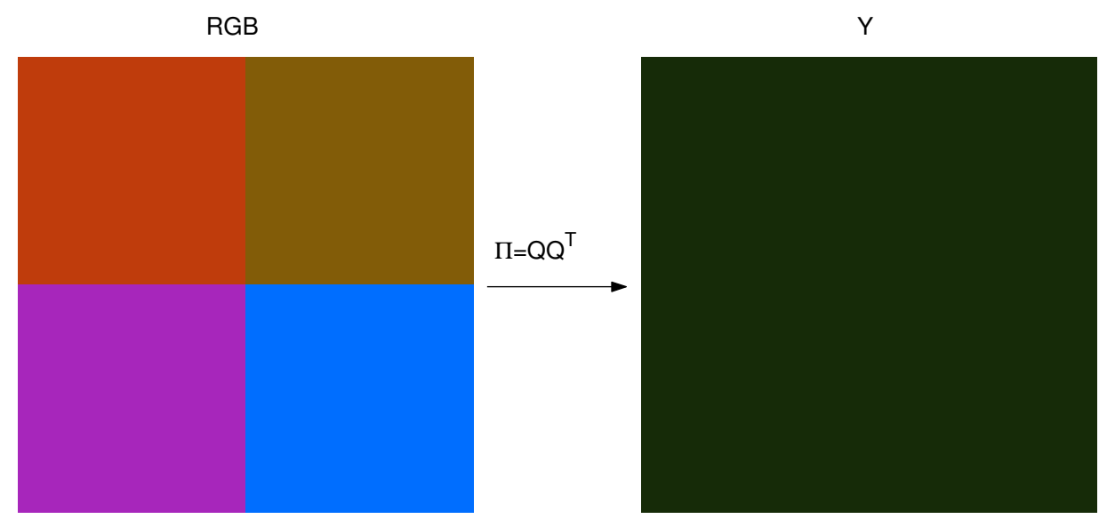
Remarks:

- The isotropic equality requirement ensures that if the point $\mathbf{x}$ is a local maximum for the corner detector then it remains a local maximum if we restrict the detector to a subspace through $\boldsymbol{x}$.
- We may want to attain efficiency of detection by using a 1D corner detector in the x-direction; we could then cull the points which are poor 1D corners and then do a full corner detector evaluation at the remaining points in the image.
- Any corner detector satisfying Axiom 1 for $d_P = n$ depends only on the eigenvalues $\lambda_1, \ldots, \lambda_n$ of A^TA.

Axiom 2

Let $\bar{Q} \in \mathbb{R}^{Nm\times Nd_Q}$ with $d_Q \leq m$ be a matrix with orthonormal columns. Then $\boxed{f(A^TA) \geq f(A^T\bar{Q}\bar{Q}^TA)}$.

Motivation: RGB → YIQ



Axiom 3

If $A_1^TA_1 \leq A_2^TA_2$ then $f(A_1^TA_1) \leq f(A_2^TA_2)$.

Motivation: The matrix A^TA provides a measure of both the strength of the intensity gradients and their independence. This can be encapsulated by the natural ordering on symmetric matrices. Thus the condition $A_1^TA_1 \leq A_2^TA_2$ means that the gradient vectors at $\mathbf{x}_2$ are stronger and/or more independent than those at $\mathbf{x}_1$ where $A_1 = A(\mathbf{x}_1)$ and $A_2 = A(\mathbf{x}_2)$.

Axiom 4

The corner detector over a set of constant eigen-energy points attains its maximum value at a point of isotropy.

Motivation: the axiom requires that the best corner (as measured by the corner detector function f) subject to the restriction of constant eigen-energy $\lambda_1^q + \cdots + \lambda_n^q = c$ for some $q > 1$ is that corner that does not have a weak direction.

	Axiom 1			Axiom 2	Axiom 3	Axiom 4
	$f(A^TA) \leq f(P^TA^TAP)$	Rotation Invariance	Isotropy Condition			
Harris-Stephens	No	Yes	No	No	No	only for $n = 2$
Modified Rohr	No	Yes	No	Yes	Yes	Yes
Förstner	Yes	Yes	No	Yes	Yes	Yes
Shi-Tomasi	Yes	Yes	Yes	Yes	Yes	Yes

Conclusions

- We presented a general framework to analyze and compare currently used corner detectors
- We extended the corner detectors in order to deal with images of different pixel and intensity dimensions
- We presented a set of four axioms that a reasonable detector based on the A^TA matrix should satisfy
- For each of the proposed axioms we outlined its theoretical and practical implications
- We studied the compliance with the axioms of each of the considered detectors

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