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Cyclic LTI Systems

Abstract

Cyclic LTI system is the LTI system where all the time indices are represented as modulo of some integer L. In this project, basic principles of cyclic multirate systems and filter banks are presented. The important differences of cyclic filter banks and traditional (noncyclic) filter banks are explained and additional freedom that cyclic system offers is introduced.

1. Introduction

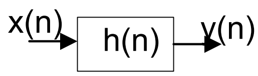


Fig.1 *Cyclic LTI system*

The system presented as block diagram in Fig.1 is a *cyclic LTI system* if the output *y(n)* is found as circular (cyclic) convolution [2] of *x(n)* and *h(n)*:

$$y(n) = \sum_{m=0}^{L-1} x(m)h(< n-m >_L), \quad 0 \le n \le L-1$$

Corresponding L-point DFTs satisfy the following relation: *Y(k) = X(k)*H(k),* $0 \le k \le L-1$. All convolutions in cyclic LTI systems are circular and denoted with cyclic(L) where L defines the range for which the input of the system is defined. Since a cyclic(L) signal can be represented by *L-DFT* coefficients instead with the entire Fourier transform, more freedom in the design is obtained and paraunitary, power complementary and linear phase properties are imposed only on discrete frequency grid and allow more relaxed conditions.

The L-point DFT for cyclic system is defined as: $H(k) = \sum_{n=0}^{L-1} h(n)W_L^{kn}$, $W_L^k = e^{\frac{-j2\pi k}{L}}$

where W_L^k is a *cyclic unit-delay* operator and generates a base of building blocks in the implementation of cyclic systems together with multipliers and adders. The unit delay operator is analogous to z^{-1} in standard DSP block diagrams.

2. Cyclic Multirate LTI Systems

2.1. Cyclic Decimators and Expanders

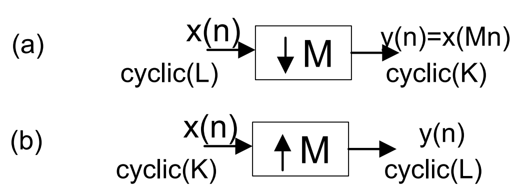


Fig2.1 *(a) M-fold decimation of cyclic(L) input*
(b) M-fold expander of cyclic(K) input

The *cyclic decimator* shown in *Fig.2.1a* has the input-output relation *y(n) = x(Mn),* where we assume that *L* is an integer multiple of *M, i.e. L = K*M.* The input x(n) and the output *y(n)* are shown in *Fig.2.2a & 2.2b* for *L=6, M=2* and it can be seen that the output is *cyclic(K)* signal.

If X(k) is the K-point DFT of x(n) and Y(K) is K-point DFT of y(n) then, by the definition, the output of the cyclic decimator is:

$$y(n)=x(Mn)=\frac{1}{L}\sum_{m=0}^{L-1}X(m)W_L^{-m(Mn)}=\frac{1}{L}\sum_{m=0}^{L-1}X(m)W_K^{-mn}=\frac{1}{L}\sum_{k=0}^{K-1}W_K^{-m}\sum_{i=0}^{M-1}X(Ki+k)=\frac{1}{K}\sum_{k=0}^{L-1}W_K^{-kn}\frac{1}{M}\sum_{i=0}^{M-1}X(Ki+k)$$

and from here it follows that:

$$Y(k)=\frac{1}{M}\sum_{i=0}^{M-1}X(k-iK),\ \ 0\le k\le K-1$$

The *cyclic expander* shown in *Fig.2.1b* has the input-output relation:

$$y(n)=\begin{cases}x(n/M), & \text{n is multiple of M}\\ 0, & \text{otherwise}\end{cases}$$

The input x(n) and the output *y(n)* are demonstrated in *Fig.2.2c and Fig.2.2d* for *L*=3, *M*=2 and it can be seen that the output is *cyclic(L*M)* signal of M-fold expander.
If X(k) is the L-point DFT of x(n) and Y(K) is L-point DFT of y(n) then, by the definition, the output of the cyclic decimator is: Y(k) = X(Mk) = X(k) for $0\le k\le L-1$. The L-point DFT Y(k) has the period *K=L/M,* since X(k) are K point DFT coefficients of x(n).

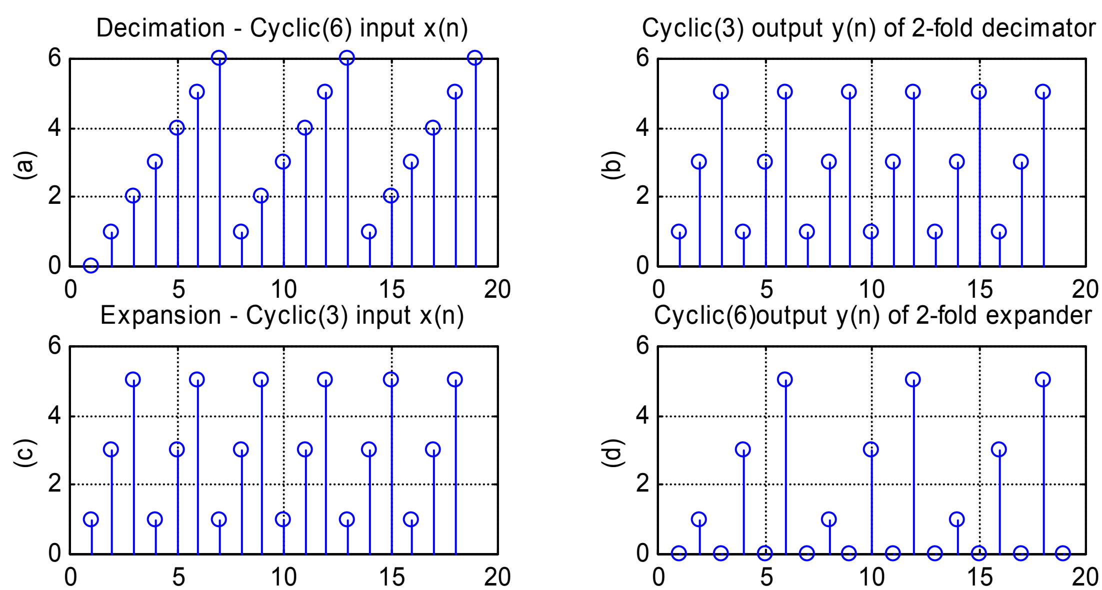


Fig2.2 *(a) Example of cyclic(L) input (b) Two-fold decimated version*
(c) Example of cyclic(L) input (d) Two-fold expanded version

```
% Cyclic decimator and expander
clear;
L =6;M=2;K=L/M;%L has to be integer multiple of M
a = eye(L);
for i=1:L-1
   a = [a,eye(L)];
end
x1 = 1:L; x=x1*a;
figure(1)
% decimation by M
yd = x(1:M:length(x));
subplot(2,2,1)
stem(x(0:length(yd)));grid;
title('Decimation - Cyclic(6) input x(n)');ylabel('(a)');
subplot(2,2,2)
stem(yd);grid;
title('Cyclic(3) output y(n) of 2-fold decimator');ylabel('(b)');
```

```
%expansion by M
N = M*length(yd);
ye = zeros(1,N);ye(1:M:N) = yd;
subplot(2,2,3)
stem(yd);grid; title('Expansion - Cyclic(3) input x(n)');ylabel('(c)');
subplot(2,2,4)
stem(ye(0:length(yd)));grid;
title(' Cyclic(6)output y(n) of 2-fold expander');ylabel('(d)');
```

2.2 Polyphase representation

The *L-point* DFT of a *cyclic(L)* impulse response *h(n)* can be expressed as

$$H(k)=\sum_{n=0}^{L-1}h(n)W_L^{nk} = \sum_{m=0}^{M-1}W_L^{Km}\sum_{n=0}^{K-1}h(Mn+m)W_K^{kn} = \sum_{m=0}^{M-1}W_L^{-km}E_m(k)$$

This equation implies that m^{th} *polyphase* component is $e_m(n)=h(Mn+m)$ and from this definition $E_m(k)$ is the K-point DFT of $e_m(n)$ and thus *cyclic(K):* $E_m(k)=\sum_{n=0}^{K-1}h(Mn+m)W_K^{kn}$, $0\le k\le L-1$.

$E_m(k)$ corresponds to $E_m(z^M)$ in noncyclic case. Similarly, the Type 2 polyphase decomposition is given by $H(k)=\sum_{m=0}^{M-1}W_L^{-km}R_m(k)$ *Cyclic polyphase* form of the M decimation filter is given in the *Fig.2.3*

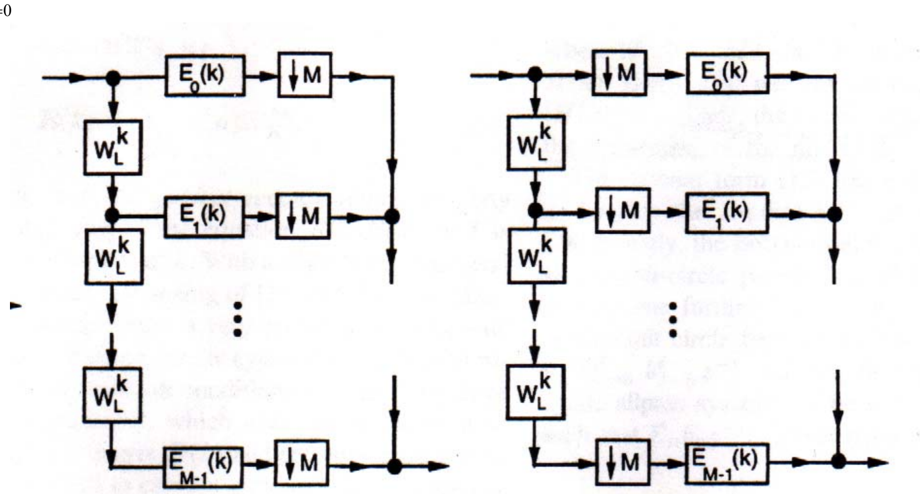


Fig 2.3*(a) Cyclic polyphase form (b) Simplification by using Noble identities*

Noble identities

The period of the polyphase components is *K=L/M* and we can use Noble identities to locate them to the right side of the decimators – similar to traditional noncyclic case. Since polyphase component is cyclic(K) that implies $E_m(k) = E_m(k+iK)$.
If U(k) is the L-point DFT of u(n) and V(k) is the K-point DFT of v(n), then from *Fig. 2.5*a and 2.5b follows:

$$V(k)=\frac{1}{M}\sum_{i=0}^{M-1}U(k+iK)=\frac{1}{M}\sum_{i=0}^{M-1}X(Ki+k)E_m(k+iK)=\frac{E_m(k)}{K}\sum_{i=0}^{M-1}X(Ki+k)$$

However, the last line shows that *V(k)* is the same as the *K-point DFT* of the system output shown in *Fig. 2.4b* is and the noble identity is established. Outputs of the system showed in *Fig 2.5b* and *Fig 2.5d* are the same so the theory behind previous derivation is consistent. In the same way, other Noble identity for expander is proved.

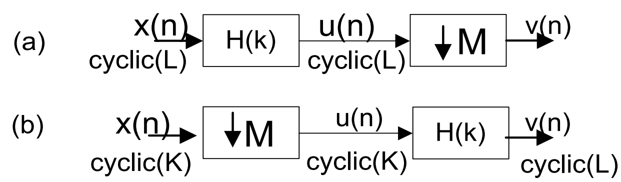


Fig2.4 (*a*) *System with cyclic(K) filter before decimator M, L=M*K*
*(b) System with cyclic(K) after decimator M, L=M*K*

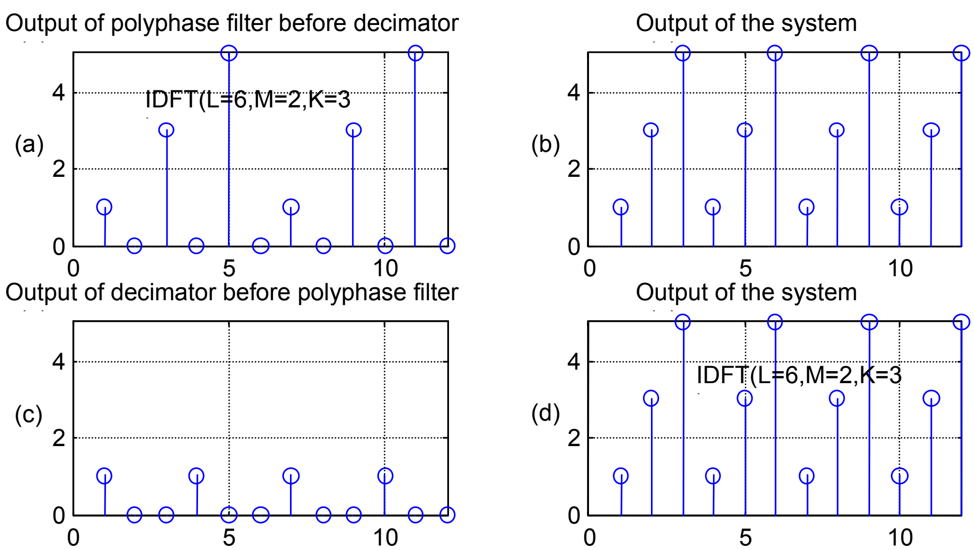


Fig 2.5 *(a) & (b)Polyphase component before decimator*
(c) & (d) Polyphase component after decimator

```
%Noble identities, M=2
x = [1 0 0 0 0 0];%impulse response
h=x1;
e0 = h(1:M:L); % corresponds to E0(z)
e0M = zeros(1,L); e0M(1:M:L) = e0; %corresponds to E0(z^M)

figure(2)
% polyphase component before decimator
u = real(ifft(fft(x,L).*fft(e0M,L),L));
v = u(1:M:length(u)); %decimation
subplot (2,2,1)
stem([u u]);grid;axis([0 12 0 5]);
title('Output of polyphase filter before decimator u(n)');
gtext('IDFT(L=6,M=2,K=3)'); ylabel('(a)');
subplot(2,2,2)
stem([v v v v]);grid;axis([0 12 0 5]);
title('Output of the system v(n)');ylabel('(b)');

%poluphase component after decimator
u = x(1:M:length(x)); %decimation
v = real(ifft(fft(x,K).*fft(e0,K),K));
subplot (2,2,3)
stem([u u u u]);grid;axis([0 12 0 5]);
ylabel('(c)');
title('Output of decimator before polyphase filter u(n)');
subplot(2,2,4)
stem([v v v v]);grid;axis([0 12 0 5]); title('Output of the system v(n)');
```

In the previous discussion same notation for the impulse response of L-point DFT and K-point DFT filters of the same filter *h(n)* component is used. In the following text, if the distinction between these two filters is not clear from the contest $H_m^{(L)}(k)$ (before decimator) and $H_m^{(L)}(k)$ (after decimator) terms are used.

2.3. M-channel cyclic(L) filter banks

M-channel cyclic(L) filter bank is shown in *Fig.6a*. The filters *hi(n)* and *fi(n)* are confined to *(a)* and their *L*-point DFTs are denoted as *Hi(k)* and *Fi(k).* With the filters *Hi(k)* represented in *Type1 polyphase* form *(Fig.*2.3b) and *Fi(k)* represented in Type 2 form, equivalent representation of *Fig.2.6a* is shown in *Fig.2.6b*, where *E(k)* and *R(k)* are the polyphase matrices of the cyclic filter bank, interpreted as *K-point DFTs.*

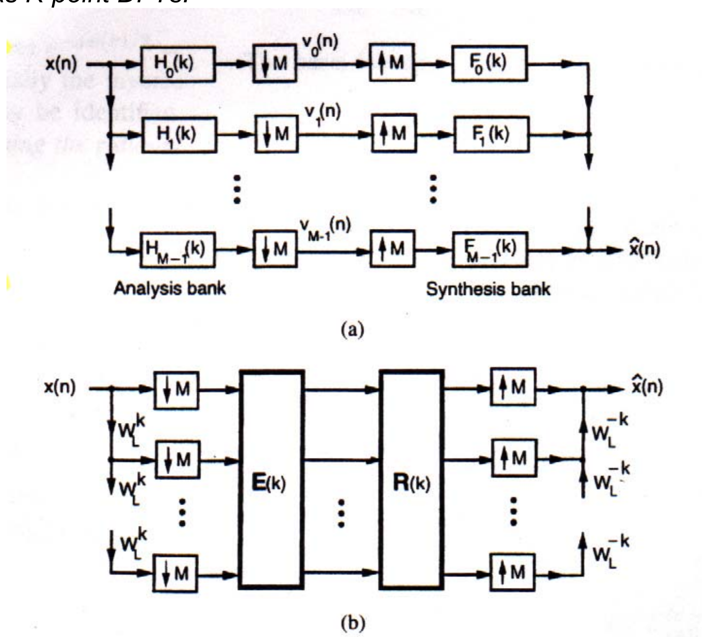


Fig 2.6 *(a) Cyclic(L) filter bank (b) Polyphase form*

Given filter bank has *perfect reconstruction* (PR) property *if and only if* the following equation is satisfied for aall the K values of *k*:

$$\mathbf{R}(k)\mathbf{E}(k) = \mathbf{I}$$

Cyclic alias-component (AC) matrix ***H**(k)* can be found using decimator and expander formulas for *L-point DFT* output signal of *M-band cyclic(L)* analysis/synthesis band:

$$\hat{X}(k) = \sum_{i=0}^{M-1} X(k+iK) \sum_{m=0}^{M-1} \frac{1}{M} H_m(k-Ki)F_m(k)$$, and we have $[\mathbf{H}(k)]_{im} = \mathbf{H}_m(k-iK)$

3. Allpass and Paraunitary Properties

A cyclic(L) allpass system is one for which $|H(k)|=1,\ 0 \le k \le L-1$. Lets form LxL circulant matrix H from the impulse response h(k):

$$H = \begin{bmatrix} h(0) & h(1) & h(2) & \dots & h(L-1) \\ h(L-1) & h(0) & h(1) & \dots & h(L-2) \\ \dots & \dots & \dots & \dots & \dots \\ h(1) & h(2) & h(3) & \dots & h(0) \end{bmatrix}$$

Circulant square matrices are normal matrices and therefore their eigenvalues are DFT coefficients of the 0th row of that matrix [1]. Moreover, they are diagonalized with DFT matrix ***W***. Applying this to matrix ***H*** we have:

$$H = \frac{W\Lambda W^+}{L}\ ,\ LI = W^+W\ ,\ \Delta = diag(H(0), H(1),..., H(L-1))\ ,\ H^+H = \frac{W\Lambda^+W^+W\Lambda W^+}{L^2} = \frac{W\Lambda^+\Lambda W^+}{L}\ .$$

If *H(k)* is an allpass filter then $I = \Lambda^+\Lambda$ and $H^+H = I$. If $H^+H = I$ then $I = \Lambda^+\Lambda$ and filter *H(k)* is allpass. We proved that cyclic allpass property is equivalent to the unitarines of the matrix ***H.***

Example 3.1:

```
% Allpass and Paraunitary Properties
L=4;
h = [0 0 0 1];
H = fft(h,4);
disp(H); disp(abs(H));
H1 = gallery('circul',h);
disp(H1); disp(H1'*H1);
```

Results:

	0	1	2	3
H(k)	1	j	-1	-j
|H(k)|	1	1	1	1

$$H1 = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix}\ ,\ H1'^*H1 = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

From this example we can see that H(k) is allpass filter and H1'*H1 is identity matrix.

E(k) is said to be *cyclic paraunitary* matrix if it is unitary for all k. The perfect reconstruction property reduces to: $R(k) = E^+(k)$ or equivalently $f_i(n) = h_i^*(-n)$ and $F_i(k) = H_i^*(k)$. This implies:

(a) unit energy property i.e. $\sum_{n=0}^{L-1} |h_i(n)|^2 = 1$ and $\sum_{n=0}^{L-1} |f_i(n)|^2 = 1$

(b) power complementary property: $\sum_{k=0}^{M-1} |H_i(k)|^2 = M$ and $\sum_{k=0}^{M-1} |F_i(k)|^2 = M$

4. Comparison of Cyclic and Noncyclic System

The non-cyclic counterpart of a *cyclic(L)* LTI system *H(k)* is defined as $H_{nc}(z) = \sum_{n=0}^{L-1} h_i(n) z^{-n}$.

This can be regarded as interpolated version in the frequency domain with H(k) representing the samples of Hnc(z) at unit circle points z=WL(-K).
If Enc(z) is a PU matrix, from previous definition it follows that E(k) is cyclic-PU because each k corresponds to special z on the unit circle. However, the converse does not hold and cyclic paraunitariness is less of constraint on the coefficient e(n) than noncyclic paraunitariness.

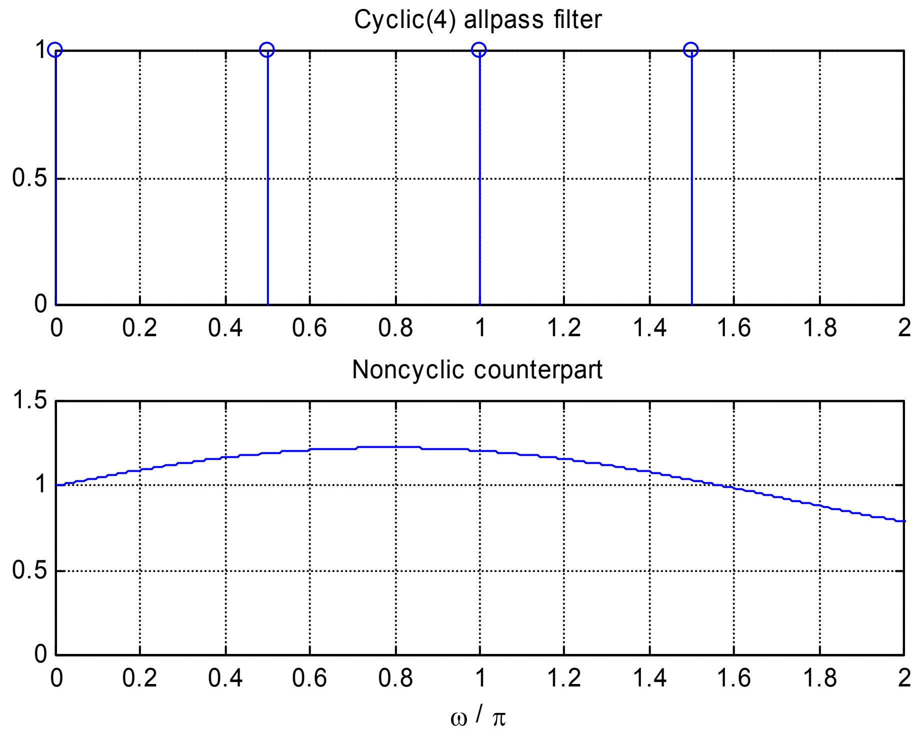
Example 4.1:

```
% Cyclic verus noncyclic systems
h = [0.5 0.5*(j-1) 0.5*j 0];
figure(3)

H = fft(h,4); disp(H); disp(abs(H));
w = [0 0.5 1 1.5];
subplot(2,1,1)
stem(w,abs(H)); grid; axis([0 2 0 1]); title('Cyclic(4) allpass filter');

[H,w] = freqz(h,1,1024,'whole');
subplot(2,1,2)
plot(w,abs(H));grid; axis([0 2 0 1.5]);
title('Noncyclic counterpart');
xlabel('\omega / \pi');
```

Result:



We can see that *H(k)* is an allpass filter in cyclic(4) sense and that noncyclic counterpart $H_{nc}(z)$ is an FIR filter but evidently not allpass.

Same reasoning can be applied to cyclic power complementary and cyclic power symmetric filters. The counterparts doesn't have to be power complementary or power symmetric and following examples show this:

Example 4.2:
Consider the cyclic(3) analysis bank:

$$H(k) = \begin{bmatrix} H_0(k) \\ H_1(k) \\ H_2(k) \end{bmatrix} = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{\sqrt{10}}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} W_3^{k} + \frac{1}{\sqrt{10}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} W_3^{2k}$$

```
% Power complementariness
h =[1 1 1; 1 -1 0; 0 -2 1]*sqrt(0.1);
H = h*dftmtx(3);
disp(diag(H'*H));
```

Result:

1 1 1

Results show that the three transfer functions satisfy cyclic power complementary property. Consider the noncyclic counterpart

$$H_{nc}(z) = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{\sqrt{10}}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} z^{-1} + \frac{1}{\sqrt{10}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} z^{-2}$$

$$\widetilde{H}_{nc} H_{nc} = 1 + 0.1z^{2} - 0.1z + 0.9z^{-1} + 0.1z^{-2}$$

It is obvious that the noncyclic counterpart is not power complementary even though the cyclic(3) system is.

Example3:

Consider a cyclic(6) tansfer function $H_0(k) = \frac{1}{\sqrt{5}}(1 + W_6^k - W_6^{2k} + W_6^{3k} + W_6^{4k})$ and let $G(k)=|H_0(k)|^2$

```
% Power symmetry
h = [1 1 -1 1 1 0]*sqrt(0.2);
H = fft(h,6);
G = (abs(H)).^2;
disp(G);
```

Result:

k	0	1	2	3	4	5
G(k)	1.8	0.2	1.8	0.2	1.8	0.2

The filter G(k) has halfband property in cyclic(6) sense and G(k) +G(K+3) = 2
Equivalently, H(k) is power symmetric. Since H(k) has a symmetric impulse response it has a linear phase i.e. $H_0(k) = \frac{W_6^{2k}}{\sqrt{5}}(-1 + 2\cos(\pi k/3) + 2\cos(2\pi k/3))$. So this filter is both linear phase and power symmetric in cyclic(6) sense. This is not possible for noncyclic FIR case.

Using the example above construction of a two channel cyclic(6) orthonormal filter bank, where the filters are nontrivial linear phase filters, is derived using the formulas:

$H_1(k) = -W_L^k H_0^*(k-K),\ K=L/2$ and $F_i(k) = H_i^*(k),\ i=1,2$.

5. Conclusion

The main purpose of this project is basic introduction to the idea of the cyclic LTI systems. Some standard problems in filter bank theory were formulated and few examples were given to illustrate the cyclic filter banks and their properties. Main advantage of cyclic systems is the extra freedom allowed for design of filter banks and it would be interesting to further investigate application of cyclic systems.

6. References

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