DESIGN OF TWO-CHANNEL LOW DELAY PERFECT RECONSTRUCTION FILTER BANKS*

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ABSTRACT

We develop a new approach for the design of two-channel perfect reconstruction (PR) filter banks that yield low system delay. The proposed design procedure is based on the idea of starting with trivial initial filters that yield low delay and satisfy the PR property. The lengths of the filters are subsequently increased in order to obtain filters with good magnitude response without sacrificing the low-delay or the PR properties of the filter bank. This is accomplished by increasing the length of the filters such that an appropriate cost function is minimized. We also develop an optimization technique to find a local minimum of the cost function.

1. INTRODUCTION

The problem of designing perfect reconstruction (PR) filter banks has been the focus of considerable research [1-4]. In all of the proposed solutions, though, the system delay has been largely ignored. However, the issue of delay is significant in the processing of temporal signals, especially for tree-structured filter banks, where stages located deep within the tree contribute exponentially to the overall system delay. Although low-delay filter banks were first investigated in [5], the filter banks thus obtained did not satisfy the PR property exactly. We develop a new approach to the design of PR filter banks. Our design procedure is based on the idea of starting with trivial filters that achieve perfect reconstruction, and subsequently updating these trivial filters to enable the design of filters of higher lengths (and thus good magnitude response) without sacrificing the perfect reconstruction property or the low-delay property. While the design procedure is illustrated, in this paper, for the case of two-channel filter banks, it can be extended to the case of filter banks with an arbitrary number of channels.

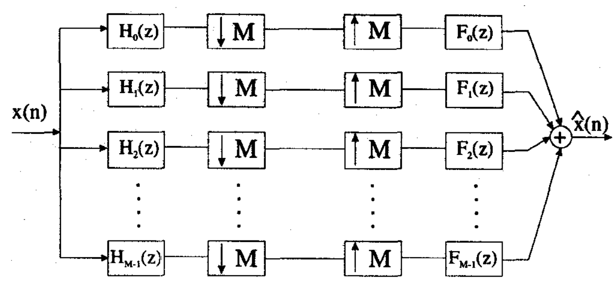

Figure 1: M-channel maximally decimated analysis and synthesis filter bank.

2. DESIGN PROCEDURE

For the typical M-channel filter bank shown in Figure 1, the condition [3] to be satisfied by the analysis and synthesis filters in order to achieve perfect reconstruction is

$$\mathbf{R}(z)\mathbf{E}(z) = \beta z^{-m_0}\mathbf{I}, \tag{1}$$

where $\mathbf{R}(z)$ and $\mathbf{E}(z)$ are the polyphase matrices of the synthesis and analysis filter banks, respectively. A necessary and sufficient condition to satisfy (1) using FIR filters is

$$\det \mathbf{E}(z) \quad = \quad \alpha z^{-K}. \tag{2}$$

There exist a number of solutions to (2), and the aim of the design procedure is to choose the solution which is optimal in some sense (typically, a criterion such as good magnitude response of the filters is used). The design procedure should also be capable of generating all of the possible solutions to (2), as opposed to generating only a subset of the solutions. This is required to ensure that the optimal solution, found from the set

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of generated solutions, is indeed the true optimum.
We approach the problem of designing PR filter banks by first specifying an initial solution to (2), and then developing a method for proceeding from this solution to the other solutions. Furthermore, we show that for certain well defined special cases, every possible solution to (2) can be obtained by this technique. In our proposed design procedure, we first choose an

$$\mathbf{E}(z) = \begin{bmatrix} k_{0,0} & k_{0,1} & \cdot & k_{0,M-1} \\ k_{1,0} & k_{1,1} & \cdot & k_{1,M-1} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ k_{M-1,0} & k_{M-1,1} & \cdot & k_{M-1,M-1} \end{bmatrix}, \quad (3)$$

as the initial solution to (2), where $k_{i,j}$ are constants, and are chosen such that $\det \mathbf{E}(z) \neq 0$. The following proposition provides a way of proceeding from this trivial solution (corresponding to filters of length M) to other solutions (corresponding to filters of length $\geq M$).

Proposition 1 *Let $\mathbf{e}_k$ be the k^{th} column of a polyphase matrix $\mathbf{E}(z)$ that satisfies det $\mathbf{E}(z) = \alpha z^{-K}$. If $\mathbf{E}'(z)$ is obtained by replacing $\mathbf{e}_k$ with*

$$\mathbf{e}_k' = z^{-r}\mathbf{e}_k + \sum_{j=0, j\neq k}^{M-1} P_j(z)\mathbf{e}_j(z), \quad (4)$$

where $P_j(z)$ are arbitrary polynomials, while retaining the remaining columns of $\mathbf{E}(z)$, then, det $\mathbf{E}'(z) = \alpha z^{-K-r}$.

Proof: The multiplication of the k^{th} column of $\mathbf{E}(z)$ by z^{-r} causes det $\mathbf{E}(z)$ to be multiplied by z^{-r}. In addition, det $\mathbf{E}(z)$ remains unchanged if a linear combination of all but the k^{th} column of $\mathbf{E}(z)$ is added to $z^{-r}\mathbf{e}_k$. This implies that det $\mathbf{E}'(z) = \alpha z^{-K-r}$.

3. DESIGN OF TWO-CHANNEL LOW-DELAY FILTER BANKS

In this section, we focus on the design of two-channel PR filter banks with minimal delay although our technique can be easily extended to the case where specific system delays are needed. To ensure both minimum delay and PR, we need to find an $\mathbf{E}(z)$ such that *det* $\mathbf{E}(z)$ is a constant.

Step 1: We use the design procedure outlined in Section 2, starting with the initial solution,

$$\mathbf{E}(z) = \begin{bmatrix} k_0 & k_1 \\ k_2 & k_3 \end{bmatrix}.$$

Step 2: We then update the first column of the polyphase matrix. The new polyphase components are given by

$$\begin{aligned} E_{00}'(z) &= E_{00}(z) + P(z)E_{01}(z), \\ E_{10}'(z) &= E_{10}(z) + P(z)E_{11}(z). \end{aligned} \quad (5)$$

Step 3: The second column is updated next and results in

$$\begin{aligned} E_{01}'(z) &= E_{00}(z) + Q(z)E_{00}'(z), \\ E_{11}'(z) &= E_{11}(z) + Q(z)E_{10}'(z). \end{aligned} \quad (6)$$

The new polyphase matrix $\mathbf{E}'(z)$ can be factored as

$$\mathbf{E}'(z) = \mathbf{E}(z) \begin{bmatrix} 1 & 0 \\ P(z) & 1 \end{bmatrix} \begin{bmatrix} 1 & Q(z) \\ 0 & 1 \end{bmatrix}, \quad (7)$$

where $\mathbf{E}(z)$ is the initial solution, and the first and second matrices correspond to the first and second updates, respectively.
To enable the generation of different solutions to (1), steps 2 and 3 above have to be repeated several times. In order to ensure that our design procedure does not exclude any solution to (2) we need to show that the factorization given by (7) captures all solutions. A general polynomial matrix with a constant determinant might not be factored using (5) and (6) alone. However, for certain well-defined cases (where the order of each element is same) it can be shown that all matrices with a constant determinant can be reduced to the factorization given in (7).

Proposition 2 *If all the elements of a polynomial matrix $\mathbf{E}(z)$, which satisfies the condition det $\mathbf{E}(z) = 1$, are of order $L-1$, then, $\mathbf{E}(z)$ can always be factored into the form*

$$\mathbf{E}(z) = \begin{bmatrix} k_0 & k_1 \\ k_2 & k_3 \end{bmatrix} \prod_{i=1}^{r} \begin{bmatrix} 1 & q_i(z) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ p_i(z) & 1 \end{bmatrix}, \quad (8)$$

Here, $p_r(z) = p_r$, $k_0k_3 - k_1k_2 = 1$ and $r \leq \lceil L/2 \rceil$.

Proof: The condition *det* $\mathbf{E}(z) = 1$ implies that

$$E_{00}(z)E_{11}(z) - E_{01}(z)E_{10}(z) = 1, \quad (9)$$

or, equivalently, that the pair $(E_{00}(z), E_{01}(z))$ is coprime. This fact can also be expressed by

$$E_{00}(z) = p_r E_{01}(z) + \epsilon_{00}(z). \quad (10)$$

From (9), it can be concluded that the pair $(E_{10}(z), E_{11}(z))$ is also coprime. Therefore,

$$E_{10}(z) = l_r E_{11}(z) + \epsilon_{10}(z), \quad (11)$$

where the order of both $\epsilon_{00}(z)$ and $\epsilon_{10}(z)$ is less than or equal $L-2$. Then, (9) can be rewritten as

$$\alpha E_{01}(z)E_{11}(z) + (\epsilon_{00}(z)E_{11}(z) - \epsilon_{10}(z)E_{01}(z)) = 1, \quad (12)$$

where $\alpha = p_r - l_r$. Both the terms on the left-hand side of (12) are polynomials in z^{-1}. However, the first term is of order $2L-2$, while the order of the second term is $2L-3$. Thus, by comparing the coefficients of $z^{-(2L-2)}$ on both sides of (12), we obtain $\alpha = 0$, or equivalently, $p_r = l_r$. This enables the factorization of $\mathbf{E}(z)$ into the form

$$\mathbf{E}(z) = \begin{bmatrix} \epsilon_{00}(z) & E_{01}(z) \\ \epsilon_{10}(z) & E_{11}(z) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ p_r & 1 \end{bmatrix}. \quad (13)$$

The determinant of the first matrix is one, which implies that the pairs $(\epsilon_{00}(z), E_{01}(z))$ and $(\epsilon_{10}(z), E_{11}(z))$ are coprime. Therefore

$$\begin{aligned} E_{01}(z) &= q_r(z)\epsilon_{00}(z) + \epsilon_{01}(z), \\ E_{11}(z) &= m_r(z)\epsilon_{10}(z) + \epsilon_{11}(z). \end{aligned} \quad (14)$$

Using a similar sequence of arguments, we can conclude that $q_r(z) = m_r(z)$. The remainder of the proof follows by recursively performing these steps until the first matrix in (13) is reduced to a constant matrix. The value of r will attain its maximum value when the order of the remainder polynomials at each stage is exactly one less than that of the divisor polynomial. Hence the maximum value of r is $\lceil L/2 \rceil$.

All minimal delay PR filter banks satisfying the assumptions of Proposition 2 can be factorized in this manner (thereby showing that our approach is capable of generating all such filter banks). However, in order to design optimal minimal delay filter banks, the order as well as the coefficients of the polynomials $p_i(z)$ and $q_i(z)$ have to be determined. The problem can be formally stated as : given the length of the filters to be designed is $2L$, find the optimal polynomials $p_i(z)$ and $q_i(z)$ in (8) that minimize the cost function

$$\begin{aligned} \phi &= \int_0^{\omega_p} (1- \mid H_0(e^{j\omega}) \mid^2)^2 d\omega + \int_{\omega_s}^{\pi} \mid H_0(e^{j\omega}) \mid^2 d\omega \\ &+ \int_0^{\omega_s} \mid H_1(e^{j\omega}) \mid^2 d\omega + \int_{\omega_p}^{\pi} (1- \mid H_1(e^{j\omega}) \mid^2)^2 d\omega \end{aligned}$$

One way to solve the problem of finding the optimal assignment of orders for $p_i(z)$ and $q_i(z)$ is by exhaustively trying all the possible orders of the polynomials $p_i(z)$ and $q_i(z)$. Not every assignment of orders for $p_i(z)$ and $q_i(z)$ is valid. From the proof of Proposition 2, two conditions have to be satisfied by the orders of $p_i(z)$ and $q_i(z)$ for a particular assignment of order to be valid

(i) The sum of the orders of all polynomials $p_i(z)$ and $q_i(z)$ should be $L-1$.

(ii) If order of $p_k(z) \neq 0$ (or $q_k(z) \neq 0$) then the orders of $p_l(z) \neq 0$ (or $q_l(z) \neq 0$) $\forall\ l > k$. Note that $p_r(z)$ is a constant.

The first condition has to be satisified because the length of the filters is $2L$. The second condition has to be satisfied since, after the first step in the proof of Proposition 2, the order of the remainder polynomial is at least 1 less than that of the divisor. Hence in the subsequent step when the divisor is divided by the remainder, the quotient is at least a first order polynomial till the polyphase matrix is reduced to a constant matrix.

As stated earlier, one way to determine the optimal $p_i(z)$ and $q_i(z)$ is to try all possible orders which satisfy conditions (i) and (ii). For a given assignment of orders, we optimize over the coefficients of the polynomials to minimize ϕ. The particular assignment of orders where the optimization of coefficients leads to the minimum value of ϕ then determines the optimal minimal delay filter bank. As the length of the filters is increased, the exhaustive search through all the possible assignment of orders becomes increasingly complex. One way of reducing the complexity of the optimization is through a suboptimal optimization procedure for finding the polynomials. This suboptimal optimization algorithm can be explained as follows

1. Start with some initial orders of $p_i(z)$ and $q_i(z)$ that satisfy conditions (i) and (ii).
2. Optimize over the coefficients of the polynomials to minimize the cost function ϕ. Call this minimum value ϕ_0.
3. Set $\phi_{min} = \phi_0$.
4. For every pair of polynomials $(s(z), t(z))$, where $s, t \in \{p_i(z), q_j(z)\}$, $1 \leq i, j \leq r$ and $s \neq t$
   (a) Increase the order of one of the polynomials by one and decrease the order of the other by one.
   (b) If the resulting assignment of orders is valid
      - Optimize over the coefficients to minimize ϕ. Call the minimum value ϕ_1,
      - If $\phi_1 < \phi_0$ then $\phi_0 = \phi_1$ otherwise undo step (a),

   otherwise, if the resulting assignment of orders is not valid, undo step (a).
5. If $\phi_0 < \phi_{min}$ go to step 3, otherwise terminate the algorithm.

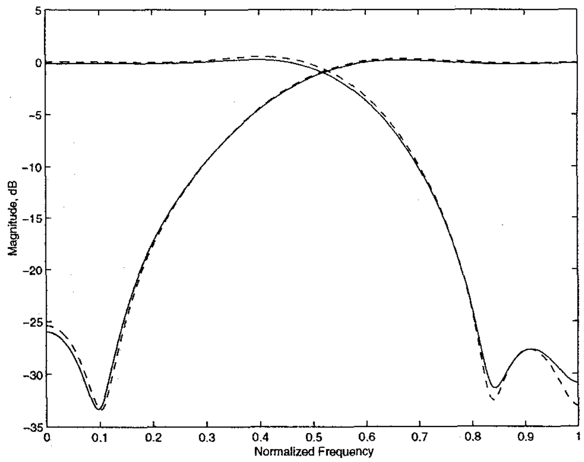

Figure 2: Magnitude response of filters (length 8, delay 1 sample) designed using the proposed technique (solid) and those designed using the technique described in [4] (dotted)

4. RESULTS

We discuss the results obtained from the application of our design procedure to two-channel low-delay PR filter banks. For this purpose, the length of the filters was chosen to be 8, and the filters were designed to obtain an overall system delay of 1 sample (which is, in fact, the minimal system delay that can be achieved with causal filters). In order to obtain the filters, we tried the exhaustive search as well as the suboptimal optimization technique developed in the previous section. In this particular case, it was found that both techniques produced the same results. For a general case, however, the suboptimal optimization will yield a local minimum as compared to the global minimum which can be found by the exhaustive search.
The magnitude response of the filters is shown in Figure 2. The figure also indicates the magnitude response of the minimal delay filters designed in [5], where perfect reconstruction was only approximated. It happens, in this case, that the magnitude response of filters designed using both approaches is similar. Although the quality of the filters obtained by both the approaches is comparable, the filters designed using our technique satisfy the perfect reconstruction property *exactly*, and not approximately. The impulse response coefficients of the two analysis filters designed using our approach are indicated in Table 1.

n	$h_0(n)$	$h_1(n)$
0	0.38747117903359	0.41245920113651
1	0.61611327689259	-0.60169737809984
2	0.13542939522761	0.10090503465235
3	-0.19927909934796	0.15862327002939
4	0.00289364270665	-0.00128280096971
5	0.06956104478971	-0.03083759287392
6	-0.01883219945646	0.00834863394468
7	-0.00822350319445	0.00364561866882

Table 1: Impulse response coefficients, $h_0(n)$ and $h_1(n)$, for filters of length 8 designed using our approach to provide a system delay of 1 sample.

5. CONCLUSION

In this paper, we present a new technique for designing perfect reconstruction filter banks. The idea behind our approach is to start with perfect reconstruction filters, of trivial length, that also possess the characteristics of the desired class of filter bank. The length of the filters is subsequently increased to improve the magnitude response of the filters, while ensuring that both the perfect reconstruction and the other initial constraints continue to remain valid.
It is also shown that, for certain well-defined cases, the design procedure is capable of generating all the possible solutions for minimal delay filter banks. We also develop a suitable optimization technique that allows us to find a solution which is the local minimum of a suitable cost function. The results that are presented show that, while the magnitude response of the low-delay filter banks designed with our approach is similar to that obtained using other techniques, the advantage with our procedure is that it satsifies the perfect reconstruction property exactly. Previous work [6] on the design of perfect reconstruction low-delay filter banks did not address the issue of completeness, whereas our research encompasses these aspects. It must be mentioned that, although the paper discusses the application of the design procedure to the case of two-channel filter banks, the technique can be easily extended to the case of filter banks with an arbitrary number of channels.

6. REFERENCES

[1] M. J. T. Smith and T. P. Barnwell III, "A Procedure for designing exact reconstruction filter banks for tree structured sub-band coders," *Proc. Int. Conf. Acoust., Speech, and Signal Process.*, San Diego, CA, pp. 27.1.1-27.1.4, March 1984.

[2] P. P. Vaidyanathan, "Theory and design of M-channel maximally decimated quadrature mirror filters with arbitrary M, having perfect reconstruction property," *IEEE Trans. Acoust. Speech and Signal Proc.,* vol. ASSP-35, pp. 476-492, April 1987.

[3] P. P. Vaidyanathan, *Multirate Systems and Filter Banks.* Englewood Cliffs, NJ: Prentice Hall, 1993.

[4] M. Vetterli, "A theory of multirate filter banks," *IEEE Trans. Acoust. Speech and Signal Proc.,* vol. ASSP-35, pp. 356-372, March 1987.

[5] K. Nayebi, T. P. Barnwell III and M. J. T. Smith, " Low delay filter banks: Design and evaluation," *IEEE Trans. Sig. Proc.,* vol. 42, pp. 24-31, Jan. 1994.

[6] G. Schuller and M. J. T. Smith, "A new algorithm for efficient low delay filter bank design", *Proc. Int. Conf. Acoust., Speech and Sig. Proc.*, vol. 2, pp. 1472-1475.