1997 IEEE International Symposium on Circuits and Systems, June 9-12, 1997, Hong Kong

ALIASING CANCELATION IN BLOCK FILTERS AND PERIODICALLY TIME VARYING SYSTEMS: A TIME-DOMAIN APPROACH*

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ABSTRACT

A block filter, in general, is a linear, periodic and time-varying (LPTV) system. Under certain conditions satisfied by the transfer function matrix of the filter, the system has an alias free operation, i.e., it behaves like a linear time-invariant (LTI) system. In this paper, we first develop a set of necessary and sufficient conditions for the alias free operation of block filters based on a time-domain analysis of the system. We then develop conditions for the alias free operation of general LPTV systems.

1. INTRODUCTION

Multirate systems, in general, are time-varying systems. However, under certain conditions, a multirate system behaves like an LTI system, i.e. has an alias free operation. In this paper, we determine conditions for the alias free operation of certain multirate systems. Multirate systems have been well analyzed in the frequency domain [1]. However, one associated problem with this approach has been that there are different techniques for the design of different kinds of filter banks like nonuniform filter banks, low delay filter banks, etc. To avoid this problem, a time-domain based approach for the analysis and design of multirate systems has been suggested [2].

We develop a set of necessary and sufficient conditions for the alias free operation of block filters, maximally decimated QMF banks, and periodically time varying systems by a time-domain analysis of these systems.

2. BLOCK DIGITAL FILTERS

Block implementation of digital filters has been proposed as a method of increasing the data throughput rate [3]. A multirate implementation of the block filter is shown in Figure 1. Because of the presence of down-samplers and up-samplers, the block filter of Figure 1 is, in general, an LPTV system with a period M. Our objective is to obtain restrictions on the transfer function matrix $\mathbf{P}(z)$ such that there is no aliasing component present at the output, i.e., to make the block filter of Figure 1 an LTI system. Now the output of each down-sampler is given by

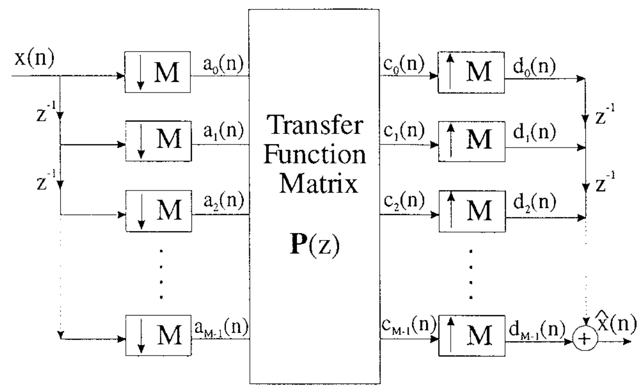

Figure 1: Block filter.

$$a_i(n) = x(nM - i). \quad (1)$$

Likewise, each output of the block filter can be written as

$$\begin{aligned} c_i(n) &= \sum_{j=0}^{M-1} \sum_{k=-\infty}^{\infty} p_{ij}(n-k) a_j(k), \\ &= \sum_{j=0}^{M-1} \sum_{k=-\infty}^{\infty} p_{ij}(n-k) x(kM-j). \quad (2) \end{aligned}$$

These are related to the output of the up-samplers by

$$d_i(n) = \begin{cases} c_i(n/M) & n = lM \quad l = 0, \pm 1 \pm 2 \ldots \\ 0 & elsewhere. \end{cases} \quad (3)$$

*This work was supported by a University of California MICRO grant with matching support from Envision Medical Corporartion and Xerox Corporation.

The final output of the block digital filter is given by

$$\hat{x}(n) = \sum_{i=0}^{M-1} d_i(n - M + 1 + i). \tag{4}$$

3. CONDITIONS FOR ALIAS FREE OPERATION OF BLOCK FILTERS

Note that, at any instant of time, the final output can be expressed as the output of just one up-sampler. To this end, we write $n = \alpha M + M - 1 - \beta$, where α is any integer and $0 \leq \beta \leq M - 1$. Then, we can rewrite Eq. (4) as

$$\hat{x}(\alpha M + M - 1 - \beta) = \sum_{i=0}^{M-1} d_i(\alpha M + i - \beta). \tag{5}$$

The value of $d_i(\alpha M + i - \beta)$ in Eq. (5)will be nonzero only if $\alpha M + i - \beta$ is divisible by M. It is easy to see that this occurs only if $i = \beta$. The final output can also be written as

$$\begin{aligned}\hat{x}(\alpha M + M - 1 - \beta) &= d_\beta(\alpha M) = c_\beta(\alpha), \\ &= \sum_{j=0}^{M-1} \sum_k p_{\beta j}(\alpha - k) x(kM - j).\end{aligned} \tag{6}$$

Let $\hat{x}_0(n)$ be the response of the system to an impulse at $k = 0$, and let $\hat{x}_r(n)$ be the response of the system to an impulse at $k = r$, then, for the system to be time-invariant we require

$$\hat{x}_r(n) = \hat{x}_0(n - r). \tag{7}$$

The problem is that for the system to be time-invariant, this condition has to be satisfied for all values of r. However, the system shown in Figure 1 is an LPTV system with a period M. This implies that $\hat{x}_{s+\gamma M}(n) = \hat{x}_s(n - \gamma M)$. We now show that, because of the periodicity of the system, we need to satisfy Eq. (7) only for $r = 0, 1, \ldots, M - 1$. For $r \geq M$ we can write $r = s + \gamma M$, where γ is any integer and $0 \leq s \leq M - 1$. However, for LTI operation, Eq. (7) implies that

$$\begin{aligned}\hat{x}_0((n - \gamma M) - s) &= \hat{x}_r(n), \\ \hat{x}_0(n - s) = \hat{x}_r(n + \gamma M) &= \hat{x}_{r-\gamma M}(n), \\ \hat{x}_0(n - s) &= \hat{x}_s(n).\end{aligned}$$

This means that, if Eq. (7) is satisfied for $r = 0, 1, \ldots, M - 1$, then, it is satisfied for all values of r. Assume $x(n) = \delta(n)$; then, the output $\hat{x}_0(n)$ is given by

$$\hat{x}_0(\alpha M + M - 1 - \beta) = p_{\beta,0}(\alpha) \quad \forall \alpha, 0 \leq \beta \leq M - 1. \tag{8}$$

Similarly, for $x(n) = \delta(n - r)$, the output $\hat{x}_r(n)$ is given by

$$\hat{x}_r(\alpha M + M - 1 - \beta) = p_{\beta,M-r}(\alpha - 1) \quad \forall \alpha, 0 \leq \beta, r \leq M - 1. \tag{9}$$

For the system to be an LTI system, we require that

$$\hat{x}_r(n) = \hat{x}_0(n - r)$$

or,

$$\begin{aligned}&\hat{x}_r(\alpha M + M - 1 - \beta) \\ &= \hat{x}_0(\alpha M + M - 1 - \beta - r), \\ &\forall \alpha, \ 0 \leq \beta, r \leq M - 1.\end{aligned} \tag{10}$$

For $\beta = 0$, LTI operation implies that

$$\hat{x}_r(\alpha M + M - 1) = \hat{x}_0(\alpha M + M - 1 - r). \tag{11}$$

Using Eqs. (8) and (9), we deduce that

$$p_{0,M-r}(\alpha - 1) = p_{r,0}(\alpha) \quad 0 \leq r \leq M - 1. \tag{12}$$

Taking z-transforms of both sides, we obtain

$$z^{-1} p_{0,M-r}(z) = p_{r,0}(z). \tag{13}$$

This equation relates the first column of the transfer function matrix $\mathbf{P}(z)$ to its first row.
For any value of $\beta = 0, 1, \ldots, M - 1$, the LTI operation of the system requires that Eq. (10) has to be satisfied. Since we have

$$\hat{x}_r(\alpha M + M - 1 - \beta) = p_{\beta,M-r}(\alpha - 1),$$

and
$\hat{x}_0(\alpha M + M - 1 - \beta - r)$

$$= \begin{cases} \hat{x}_0(\alpha M + M - 1 - (\beta + r)) & \text{if r} \leq M - 1 - \beta \\ \hat{x}_0((\alpha - 1)M + M - 1 - (\beta + r - M)) & \text{if r} > M - 1 - \beta \end{cases}$$

$$= \begin{cases} p_{r+\beta,0}(\alpha) & \text{if r} \leq M - 1 - \beta \\ p_{r+\beta-M,0}(\alpha - 1) & \text{if r} > M - 1 - \beta, \end{cases}$$

Eq. (10) leads to
$p_{\beta,M-r}(\alpha - 1)$

$$= \begin{cases} p_{r+\beta,0}(\alpha) & \text{if r} \leq M - 1 - \beta \\ p_{r+\beta-M,0}(\alpha - 1) & \text{if r} > M - 1 - \beta. \end{cases}$$

Replacing $M-r$ by r, and taking the z-transform of both sides, we obtain

$$z^{-1}p_{\beta,r}(z) = \begin{cases} p_{\beta+M-r,0}(z) & \text{if } \beta \le r-1 \\ z^{-1}p_{\beta-r,0}(z) & \text{if } \beta > r-1. \end{cases} \quad (14)$$

This relates the rth column of the transfer function matrix $\mathbf{P}(z)$ to its first column. Thus, by substituting different values of $r = 1, 2, \ldots, M-1$, we can relate all the columns of $\mathbf{P}(z)$ to its first column. Let the first row of $\mathbf{P}(z)$ be $[H_0(z), H_1(z), \ldots, H_{M-1}(z)]$. Hence, the first two columns of the matrix $\mathbf{P}(z)$ are given by $[H_0(z),\ z^{-1}H_{M-1}(z),\ z^{-1}H_{M-2}(z), \ldots, z^{-1}H_1(z)]^T$, and $[H_1(z), H_0(z), z^{-1}H_{M-1}(z), \ldots, z^{-1}H_2(z)]^T$, respectively, and so on. The general structure of the transfer function matrix under aliasing cancelation is, therefore,

$$\mathbf{P}(z) = \begin{bmatrix} H_0(z) & H_1(z) & . & H_{M-1}(z) \\ z^{-1}H_{M-1}(z) & H_0(z) & . & H_{M-2}(z) \\ z^{-1}H_{M-2}(z) & z^{-1}H_{M-1}(z) & . & H_{M-3}(z) \\ . & . & . & . \\ . & . & . & . \\ z^{-1}H_1(z) & z^{-1}H_2(z) & . & H_0(z) \end{bmatrix}. \quad (15)$$

As can be seen from the structure of the matrix, each column is obtained by shifting the previous column downward by one, and then recirculating the spilled element after multiplying it by z. Summarizing, we conclude that, for aliasing cancelation in a block filter, its transfer function matrix $\mathbf{P}(z)$ must be of the form of Eq. (15), which has been called a pseudocirculant matrix [1],[4].

The above result can be trivially extended to the case of M-band maximally decimated QMF banks. In this case, we can carry out a polyphase decomposition of the analysis and synthesis filters. The polyphase blocks can then be moved to the center by making use of noble identities. This reduces an M-band maximally decimated QMF bank to the system shown in Figure 1, with $\mathbf{P}(z) = \mathbf{R}(z)\mathbf{E}(z)$, where $\mathbf{E}(z)$ and $\mathbf{R}(\mathbf{z})$ are the polyphase matrices of the analysis and synthesis filters respectively. Thus, an M-band maximally decimated QMF bank is free from aliasing if and only if the product $\mathbf{R}(z)\mathbf{E}(z)$ is a pseudocirculant matrix.

4. ALIASING CANCELATION IN GENERAL LPTV SYSTEMS

A linear periodic time varying system with a period M, and with identical input and output rates, is characterized by a set of M transfer functions, $\{A_n(z)\}$. The system can be represented by the structure shown in Figure 2. The output of this sytem at time n is equal to the output of the filter $A_\gamma(z)$ at time $n-\gamma$, where $\gamma = n \bmod M$. Consider the LPTV system shown in Figure 2. Here, the relations between different variables are given by

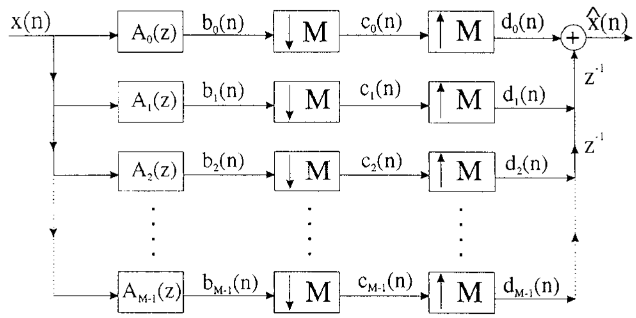

Figure 2: A linear periodic time varying system.

$$b_i(n) = \sum_k a_i(k)x(n-k),$$

$$c_i(n) = \sum_k a_i(k)x(nM-k),$$

and

$$d_i(n) = \begin{cases} c_i(n/M) & n = lM \quad l = 0, \pm 1 \ldots \\ 0 & elsewhere. \end{cases}$$

The output of the entire system can be written as

$$\hat{x}(n) = \sum_{i=0}^{M-1} d_i(n-i).$$

Let $n = \alpha M + \beta$. Again, at any instant of time, the system output will be equal to the output of only one of the up-samplers. Hence

$$\hat{x}(n) = \hat{x}(\alpha M + \beta) = \sum_{i=0}^{M-1} d_i(\alpha M + \beta - i)$$
$$= d_\beta(\alpha M) = \sum_k a_\beta(k)x(\alpha M - k). \quad (16)$$

Let $\hat{x}_0(n)$ represent the output of the system for an impulse at $k=0$. Similarly, let $\hat{x}_r(n)$ be the output corresponding for an impulse at $k=r$. For aliasing cancelation, we require that $\hat{x}_r(n) = \hat{x}_0(n-r)$, but only for $r = 0, 1, \ldots M-1$, on account of the periodicity of the system. Substituting $x(n) = \delta(n-r)$ in Eq. (16) yields

$$\hat{x}_r(n) = a_\beta(\alpha M - r). \quad (17)$$

and

$$\hat{x}_0(n-r) = \sum_{i=0}^{M-1} d_i(\alpha M + \beta - r - i)$$
$$= \begin{cases} d_{\beta-r}(\alpha M) & \text{if } \beta \geq r \\ d_{M+\beta-r}((\alpha-1)M) & \text{if } \beta < r \end{cases}$$
$$= \begin{cases} c_{\beta-r}(\alpha) & \text{if } \beta \geq r \\ c_{M+\beta-r}(\alpha-1) & \text{if } \beta < r. \end{cases}$$

Using the above equation and Eq. (17), it follows that

$$a_\beta(\alpha M - r) = \begin{cases} a_{\beta-r}(\alpha M) & \text{if } \beta \geq r \\ a_{\beta+M-r}((\alpha-1)M) & \text{if } \beta < r. \end{cases} \quad (18)$$

For $\beta = 0$, these equations reduce to

$$a_0(\alpha M - 1) = a_{M-1}(\alpha M - M), \quad \text{for } r = 1$$
$$a_0(\alpha M - 2) = a_{M-2}(\alpha M - M), \quad \text{for } r = 2$$
$$\vdots$$
$$a_0(\alpha M - M + 1) = a_1(\alpha M - M). \text{ for } r = M - 1$$

Taking z-transforms of both the sides, we obtain

$$[A_{0,0}(z),\ A_{0,1}(z),\ \ldots, A_{0,M-1}(z)] = [A_{0,0}(z),\ A_{1,0}(z),\ \ldots A_{M-1,0}(z)].$$

where $A_{0,0}(z), A_{0,1}(z) \ldots, A_{0,M-1}(z)$ represent polyphase components of $A_0(z)$. Repeating the above procedure for $\beta = 1$ yields

$$[A_{1,0}(z),\ A_{1,1}(z),\ \ldots, A_{1,M-1}(z)] = [A_{1,0}(z),\ A_{2,0}(z),\ \ldots, zA_{0,0}(z)].$$

Thus, the second row of the polyphase matrix can be obtained from the first row by shifting the first row towards the left, and then recirculating the spilled element after multiplying it by z. For different values of β, we can show that each row of the polyphase matrix is obtained from the previous row by following the procedure described above. Hence, the polyphase matrix can be written as

$$\mathbf{E}(z) = \begin{bmatrix} A_{0,0}(z) & A_{1,0}(z) & . & A_{M-1,0}(z) \\ A_{1,0}(z) & A_{2,0}(z) & . & zA_{0,0}(z) \\ . & . & . & . \\ . & . & . & . \\ A_{M-1,0}(z) & zA_{0,0}(z) & . & zA_{M-2,0}(z) \end{bmatrix}$$

It can be shown that $\mathbf{E}(z)$ has the structure of a pseudocirculant matrix. This implies that, for aliasing cancelation in LPTV systems of the type shown in Figure 2, we require that the polyphase matrix of the filters $\{A_n(z)\}$ be a pseudocirculant matrix.

5. CONCLUDING REMARKS

In this paper, we have derived conditions for the alias free operation of certain multirate systems by analyzing the system in the time-domain. In particular, for alias free operation of a block filter, we establish that its transfer function matrix should be a pseudocirculant matrix. This has also been observed in [4]. However, the main difference is that the analysis in [4] was done in frequency domain whereas we have obtained the same result with analysis in the time-domain. In this way, we have extended the time-domain analysis technique by showing that certain results that were obtained by a frequency domain analysis of multirate systems, can also be obtained in the time-domain. This technique for handling aliasing in the time-domain can be very useful in designing maximally decimated QMF banks in the time-domain. Instead of finding the coefficients of both the analysis and synthesis filters by an unconstrained optimization, as done in [2], we can use the aliasing cancelation conditions to constrain the coefficients of the analysis and synthesis filters. The filter coefficients can then be obtained by a constrained optimization of a suitable cost function. In this way, we can completely cancel aliasing in a QMF bank whereas the technique used in [2] merely minimizes aliasing. Also, for the case of a general LPTV system, we have shown that, if the polyphase matrix of the filters is a pseudocirculant matrix, then, the system behaves like a linear time invariant system.

6. REFERENCES

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