Image Segmentation via Functionals Based On Boundary Functions

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ABSTRACT

A general variational framework for image approximation and segmentation is introduced in which the boundary function has a simple explicit form in terms of the approximation function. At the same time, this variational framework is general enough to include the most commonly used objective functions. Since the optimal boundary function, that minimizes the associated objective functional for a given approximation function, can be found explicitly, the objective functional can be expressed in a reduced form that depends only on the approximating function. From this a partial differential equation descent method, aimed at minimizing the objective functional, is derived. The method is fast and produces excellent results as illustrated by a number of real and synthetic image problems.

1 INTRODUCTION

In this paper a general variational framework is presented for image segmentation and approximation. In addition to several new results, one of the main contributions is in simplifying and systematizing approaches that had previously been considered separately, especially those with Mumford-Shah objective functionals [8],[9] and those considered by Geman and others [4],[5],[6]. The common framework for these approaches also makes it much easier to do comparative studies of competing systems.

To set the stage, suppose that we are given a blurred image g over a domain Ω:

$$g = Au_0 + \eta \tag{1}$$

where A is the blurring operator, u_0 is the unblurred image and η is the noise. One approach to segmenting and approximating such an image consists of finding an approximation u and a boundary set K that minimizes an objective functional of the form

$$E(u, K) = w_1 \int_{\Omega \backslash K} (Au - g)^2 + w_2 \int_{\Omega \backslash K} \|\nabla u\|^2 + w_3 \int_K d\sigma \tag{2}$$

where the last integral term corresponds to the length of the boundary. The scalars w_1, w_2, and w_3 are weighting factors that determine respectively how closely Au approximates g, the smoothness of u and the extent of the boundary. Without loss of generality we may assume that $w_3 = 1$. Functionals of this type are often referred to as a Mumford-Shah (MS) functionals. See [7] p.24, [8], and [9] for details.

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Unfortunately numerical procedures for minimizing the MS functional encounter bookkeeping problems associated with tracking regions and their boundaries. These problems can be traced to the binary nature of the boundary description as embodied in the boundary characteristic function, which takes on the value 1 on the boundary K and 0 elsewhere. Binary descriptions of boundaries may be appropriate in some special cases but for most problems the transitions between regions can occur over several pixels rather than abruptly. Moreover the mathematical view of the boundary as the differential of a region underscores the inherent sensitivity of the boundary description process; this is entirely analogous to the sensitivity of derivatives with respect to noise.

For these reasons, it often is appropriate to specify boundaries with a function B taking continuous values between 0 and 1. Such a function might be viewed as a probability boundary description but we do not explore that issue. Instead our main concerns are utility and ease of numerical computation.

To accommodate a continuous boundary function B, the MS functional could be recast as

$$(u, B) = w_1 \int_\Omega (Au - g)^2 (1 - B)^2 + w_2 \int_\Omega \|\nabla u\|^2 (1 - B)^2 + w_3 \int_\Omega B^2 \tag{3}$$

Here we have replaced the integrals over $\Omega \backslash K$ by integrals over Ω with integrands multiplied by $(1 - B)^2$, the idea being that since $B \approx 1$ near K, the integration of terms times $(1 - B)^2$ over K is nearly 0. Similarly the boundary length integral has been replaced by the integral of B^2.

The rest of this paper focuses on minimizing a wide class of objective functionals that includes (3) as well as functionals of the type considered by Geman and others. Aside from the simplifications that result from using a common theoretical framework to compare competing methods, the main contributions of this paper can be summarized as:

1. A closed form solution is derived for the optimal boundary function B associated with a given approximation function u. The explicit form of the boundary function can then be used to reduce the objective function to a form in which the minimization problem can be recast as an equivalent PDE.

2. The general framework used in this paper allows one to compare different objective functions of the MS or Geman type for least squares approximation (L_2 norm) or the total variation (L_1 norm) approach of Osher and Rudin [10],[11].

3. A PDE descent method is used that is significantly faster than stochastic search algorithms, and as the experimental results indicate, the results compare favorably with existing methods.

4. The extent of the boundary is influenced by the terms $(1-B)^2$ and B^2 in the objective functional. One could also work with $(1-B)^{\gamma}$ and B^{γ} for any value $\gamma > 1$. However, the case $\gamma = 2$ is sufficiently general because there is an equivalence between the case of arbitrary $\gamma > 1$ and the case $\gamma = 2$ with a modified residual function [2].

In the next section we briefly review the related work in the literature, followed by a discussion of different objective functionals. This is followed by a short description of the numerical implementation and experimental results.

2 REDUCIBLE OBJECTIVE FUNCTIONALS

2.1 Relation to Related Work

There is a significant amount of related work in image processing and vision. Early work in this area dealt with scale space decompositions induced by Gaussian smoothing operators and the motion of edges (as identified with zero-crossings of the Laplacian) in scale space.

Identifying spatial discontinuities is helpful in many applications such as segmentation, optical flow, stereo, and image reconstruction. The concept of a *line process* is useful in studying these problems as one of regularization. The binary line process was introduced by Geman and Geman [4] where the authors considered simulated annealing based algorithms for achieving the global optimization. Since then several modifications of the original scheme have been suggested. Blake and Zisserman [1] formulated the same problem as minimizing an objective functional which enforces smoothness while eliminating the binary line process. See also [3], [6], [12]. Some of these recent works involve analog or continuous line processes.

Common to all these algorithms is an objective functional that:

(a) enforces closeness to the original data by including terms such as $(Au-g)^2$.

(b) promotes local smoothness away from edges by including terms depending on $\|\nabla u\|$.

(c) limits the extent of the boundary.

2.2 A General Framework

Consider now the following generalized form of the Mumford-Shah (MS) functional (3)

$$E(u, B) = \int_{\Omega} r(1-B)^2 + B^2 \qquad (4)$$

where the residual term r depends on $Au-g$ as well as ∇u. For our purposes we have found the following form of r to be most useful

$$r = w_1(Au-g)^2 + w_2\|\nabla u\| \qquad (5)$$

but more general forms of r are also considered below. Functionals of this type have the big advantage that the optimal boundary function B can be found explicitly for any nonnegative residual function r: independent of the form of r we can show that, for a given function u, the function B that minimizes $E(u, B)$ is given by

$$B = \frac{r}{1+r} \qquad (6)$$

We denote this optimal boundary function by $B = B(u)$. This allows us to eliminate B from the objective functional and (after some simple algebra) we are led to the equivalent problem of minimizing the functional $E(u) \equiv E(u, B(u))$

$$E(u) = \int_{\Omega} \frac{r}{1+r} \qquad (7)$$

It is interesting that this reduced functional is equal to the L_1 norm of the optimal boundary function B; that is minimization of the reduced functional is really the same as minimizing the L_1 norm of B subject to $B = r/(1+r)$.

2.3 Geman Type Functionals

A similar reduction procedure is possible with functionals of the type considered by Geman and others [4],[5],[6]. In [6], Geman and Reynolds looked at objective functionals depending only on the approximation function u of the form

$$H(u) = \sum_{S} \lambda(Au-g)^2 + \sum_{C} \phi(D_C(u)/\Delta) \qquad (8)$$

where λ and Δ are weights, S is the set of pixels indices, C is the set of *cliques* (or neighboring pixels), D_C is a difference function akin to a directional gradient, and ϕ is a specified function. For example, in [6], Geman and Reynolds used

$$\phi(x) = \frac{-1}{1+|x|} \qquad (9)$$

since this function was empirically noted to have "yielded consistently good results."

One can recast this type of a functional, referred to as the Geman type functional in the following discussion, in the form

$$G(u,B) = \int_\Omega w_1 (Au-g)^2 + \int_\Omega w_2 \|\nabla u\| (1-B)^2 + \int_\Omega B^2 \quad (10)$$

As in the case of MS functionals, the optimal boundary function is given by $B = r/(1+r)$ with $r = w_2\|\nabla u\|$.

The connection between this functional and (8) can be seen by substituting $B(u)$ into G to get

$$G(u,B) = \int_\Omega w_1 (Au-g)^2 + \int_\Omega \phi(w_2\|\nabla u\|) + C \quad (11)$$

where the constant $C = \int_\Omega 1$ and $\phi(x) = -1/(1+|x|)$. Thus this functional differs only by a constant from the integral form of the functional (8) for the choice $\phi(x) = -1/(1+|x|)$. (Note the slight difference however in that our reformulation of the Geman functional uses ∇u to measure approximation smoothness rather than separate terms for $\partial u/\partial x$ and $\partial u/\partial y$.)

Explicit formulas for the optimal boundary term make it easier to gain insight into the expected behavior of the segmentation algorithm. For example, the optimal boundary functions for the MS functional (4) and the Geman functional (8) are both given by $B = r/(1+r)$ where

$$r_{MS} = w_1 (Au-g)^2 + w_2\|\nabla u\| \qquad \text{Mumford-Shah}$$

$$r = w_2\|\nabla u\| \qquad \text{Geman}$$

From this we see that the Geman optimal boundary function does not include the approximation error term $(Au-g)^2$. That is, it derives all of its boundary information from the gradient of the approximation function u. Since the term $(Au-g)^2$ measures the "residual noise" in the approximation we expect that the Geman boundary function should be smoother than the MS boundary function. We also see that, for the same function u and the same gradient weight w_2, the MS residual r_{MS} is larger than the Geman residual r_G. Consequently, $B_{MS} \geq B_G$.

However, the increased smoothness of B_G over B_{MS} and the inequality $B_{MS} \geq B_G$ assumes that the same approximation u used in determining r_{MS} and r_G. In general it is not the case that the optimal approximation function u is the same for the MS and Geman functionals.

3 Numerical Implementation

The previous sections discussed various functionals for approximating and segmenting images. These functionals contain terms related to the approximation error and the smoothness of the approximation as well as the extent of the boundary. Once the form of the functional has been selected, the nontrivial problem of finding the minimizing approximation u has to be addressed. Typically the desired approximation is an equilibrium solution of a nonlinear diffusion PDE with certain boundary conditions. A general procedure for finding these PDEs is outlined in [2].

To illustrate, suppose that we wanted to minimize a functional of the form

$$E(g,u) = \int_\Omega (u-g)^2 + \nabla u \cdot \nabla u \quad (12)$$

where g is the given image and u is an approximation of g. The minimizing approximation u for this functional satisfies the ellpitic equilibrium PDE

$$\Delta u = u - g \quad (13)$$

$$\partial u/\partial n = 0 \text{ on } \partial\Omega \quad (14)$$

where Δu is the Laplacian of u and $\partial u/\partial n$ denotes the normal derivative. Numerically we can either solve for the equilibrium solution directly or follow u as a function of t from an initial approximation, such as $u_0 = g$, by integrating the diffusion PDE

$$u_t = g - u + \Delta u \quad (15)$$

subject to the Neumann boundary condition (14). Starting from the initial condition u_0 the image u evolves as $t \to \infty$ toward the equilibrium solution. A similar procedure applies for the reduced form of MS functionals or the Geman type functionals (see[2] for details.)

Once we have defined a residual function and obtained the corresponding PDE for u, we then use Euler's method to integrate the descent PDE and halt the integration when the decrease in the value of the objective functional becomes less than a user supplied tolerance (typically, 1-5% in our experiments).

4 Experimental Results and Conclusions

Figure 1(a) presents a piecewise constant image similar to an example considered by Richardson in [13], but with additive Gaussian noise. Figures 1(b) and 1(c) show respectively the approximation function u and the boundary function b for the MS functional. It is interesting that the associated Geman approximation function u for the same parameter choices is nearly identical as seen in Figure 1(d), but the Geman boundary function in Figure 1(e) is not as noisy as the MS boundary function.

There is also another approach, involving iteration, that can be used to remove noise in both the approximation and the boundary function. In this approach, we start with an initial image g and generate an approximation u. In turn this approximation is used as an initial image to generate another approximation u_1, and we may repeat as often as desired with each successive approximation becoming smoother. This is illustrated in Figure 1(f) which shows the first iterate u_1 for the noisy piecewise constant example image. Figure 1(g) shows the associated boundary function which is almost entirely noise-free.

An example on a face image is presented in Figure 2. The approximation function u in the figure appears much like an artists' sketch. This simplification means that the approximations are more suitable for face feature extraction than the original images. Examples of further applications to segmenting medical and satellite images can be found in [2].

4.1 Conclusions

We have presented a general framework for segmenting images and obtaining region boundaries based on minimizing an objective functional for which the optimal boundary function has a particularly simple form. A PDE descent procedure can be used to minimize the reduced form of the objective function. Many commonly used segmentation approaches can be represented in this framework, which is also general enough to include least squares and total variation forms.

Further research is needed on the problem of selecting the best weights for a given image or class of images, as well the problem of automatically selecting the best choice of norms for the residual function.

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5 REFERENCES

[1] A. Blake and A. Zisserman, "Visual Reconstruction," MIT Press, Cambridge, MA, 1987.

[2] G. Hewer, C. Kenney, and B. S. Manjunath, "Variational Image Segmentation using Boundary Functionals," UCSB ECE Technical Report TR#96-07, April, 1996.

[3] D. Geiger and F. Girosi, "Parallel and deterministic algorithms from MRFs: Surface reconstruction," *IEEE Trans. Pat. Anal. Mach. Intell.*, 13(5), pp. 401-412, May 1991.

[4] S. Geman and D. Geman, "Stochastic relaxation, Gibbs distributions, and Bayesian restoration of images," *IEEE Trans. Pat. Anal. Mach. Intell.*, 6, pp. 721-741, 1984.

[5] D. Geman, S. Geman, C. Graffigne and P. Dong, "Boundary detection by constrained optimization," *IEEE Trans. Pat. Anal. Mach. Intell.*, 12, pp. 609-628, 1990.

[6] D. Geman and G. Reynolds, "Constrained restoration and the recovery of discontinuities," *IEEE Trans. Pat. Anal. Mach. Intell.*, 14, pp. 376-383, March 1992.

[7] J. Morel and S. Solimini, *Variational Methods in Image Segmentation*, Birkhauser, Boston, 1995.

[8] D. Mumford and J. Shah, "Boundary detection by minimizing functionals," *IEEE Conference on Computer Vision and Pattern Recognition*, San Francisco, 1985.

[9] D. Mumford and J. Shah, "Optimal approximation by piecewise smooth functions and associated variational problems," *Comm. on Pure and Appl. Math.*,Vol. XLII, No. 4, 1989.

[10] S. Osher and L. Rudin, "Feature-oriented image enhancement using shock filters," *SIAM J. Numer. Anal.*, vol. 27, pp. 919-940, 1990.

[11] S. Osher and L. Rudin, "Shocks and other nonlinear filtering applied to image processing," *Proceedings SPIE Appl. Dig. Image Proc. XIV*, vol. 1567, pp. 414-430, 1991.

[12] A. Rangarajan and R. Chellappa, "A continuation method for image estimation using the adiabatic approximation," in R. Chellappa and A. K. Jain, editors, *Markov Random Fields: Theory and Applications*, Academic Press, 1993.

[13] T. Richardson and S. Mitter, "Approximation, computation and distortion in the variational formulation," in *Geometry-Driven Diffusion in Computer Vision*, B.M. ter Harr Romeny, Ed., Kluwer Academic Publ., Boston, 1994, pp. 169-190.

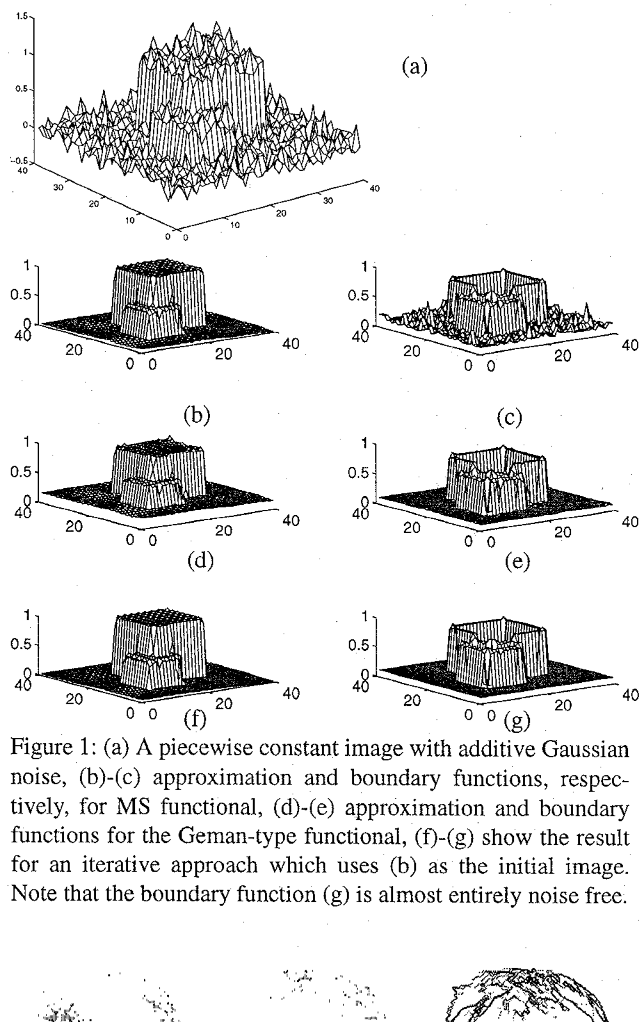

Figure 1: (a) A piecewise constant image with additive Gaussian noise, (b)-(c) approximation and boundary functions, respectively, for MS functional, (d)-(e) approximation and boundary functions for the Geman-type functional, (f)-(g) show the result for an iterative approach which uses (b) as the initial image. Note that the boundary function (g) is almost entirely noise free.

Figure 2: Smoothing face images. Original image is on the left, approximation in the middle and the boundary function is shown on the right.