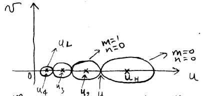
Derivation of the Gabor Filter Dictionary paramters

Given U_H, U_L, K, S, Find a, σ_u, σ_v



$g_{mn}(x,y) = a^{-m} g(x', y')$, $x' = a^{-m}(x\cos\theta + y\sin\theta)$

$y' = a^{-m}(-x\sin\theta + y\cos\theta)$

$\theta = \frac{n\pi}{K}$

$G_{mn}(u,v) = a^m G\left(a^m(u\cos\theta + v\sin\theta), a^m(-u\sin\theta + v\cos\theta)\right)$

- $G_{00}(u,v) = \exp -\frac{1}{2}\left(\frac{(u-u_H)^2}{\sigma_u^2} + \frac{v^2}{\sigma_v^2}\right)$

$G_{10}(u,v) = a \exp -\frac{1}{2}\left(\frac{(au - u_H)^2}{\sigma_u^2} + \frac{(av)^2}{\sigma_v^2}\right)$

- we want magnitude $G_{00}(u,v)$ & $G_{10}(u,v)$ to touch at halfpeak

$\therefore \exp -\frac{1}{2}\left(\frac{(u_1 - u_H)^2}{\sigma_u^2} + 0\right) = \frac{1}{2}$ — ①

& $a \exp -\frac{1}{2}\left(\frac{(au_1 - u_H)^2}{\sigma_u^2} + 0\right) = \frac{a}{2}$ — ②

① ⇒ $u_1 = -\sigma_u\sqrt{2\ln 2} + u_H$

② ⇒ $u_1 = \frac{u_H}{a} + \frac{\sigma_u}{a}\sqrt{2\ln 2}$

$\Rightarrow \boxed{\sigma_u = \frac{(a-1)\,u_H}{(a+1)\sqrt{2\ln 2}}}$

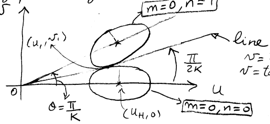
& ② gives ellipse eqn with center $u_2 = \frac{u_H}{a}$

For $m=2, n=0 \Rightarrow u_3 = \frac{u_H}{a^2}$ $\left(G_{20}(u,v) = a^2 \exp -\frac{1}{2}\left[\frac{(a^2u - u_H)^2}{\sigma_u^2} + \frac{v^2}{\sigma_v^2}\right]\right)$

Hence, for $m = S-1 \Rightarrow u_{center} = \frac{u_H}{a^{S-1}}$ which equals u_L

$\Rightarrow \frac{u_H}{a^{S-1}} = u_L \Rightarrow \boxed{a = \left(\frac{u_H}{u_L}\right)^{\frac{1}{S-1}}}$

* For the rotated ellipses at $\theta = \frac{n\pi}{K}$, they have to touch also at half peak magnitude:



we need the two ellipses to touch at half peak, which is the same as the line $v = \tan\frac{\pi}{2K}\, u$ being tangent to the ellipse $G_{00}(u,v) = \exp -\frac{1}{2}\left(\frac{(u-u_H)^2}{\sigma_u^2} + \frac{v^2}{\sigma_v^2}\right) = 1/2$

i.e. intersecting $v = mu$

with $\dfrac{(u-u_H)^2}{\sigma_u^2\, 2\ln 2} + \dfrac{v^2}{\sigma_v^2\, 2\ln 2} = 1$ } for a single pt (u_1, v_1)

by eliminating "v" from the second eqn, and setting the discriminator to zero for a unique solution in the eqn of "u" one gets the following eqn:

$$4 u_H^2 b^4 = 4(b^2 + a^2 m^2)(b^2 u_H^2 - a^2 b^2)$$

where $b^2 = \sigma_v^2 (2\ln 2)$, $a^2 = \sigma_u^2 (2\ln 2)$, $m = \tan\frac{\pi}{2K}$

Hence

$$\sigma_v = \frac{\tan(\pi/2K)}{\sqrt{2\ln 2}} \sqrt{u_H^2 - \sigma_u^2 (2\ln 2)}$$

$$\Rightarrow \boxed{\sigma_v = \tan(\pi/2K) \sqrt{\frac{u_H^2}{2\ln 2} - \sigma_u^2}} \quad \# \quad \text{—} \; 3$$

which is the same as the eqn mentioned in the paper after simplification

In the paper, we have

$$\sigma_v = \frac{\tan\left(\frac{\pi}{2K}\right)\left(u_H - (2\ln 2)\frac{\sigma_u^2}{u_H}\right)}{\sqrt{2\ln 2 - (2\ln 2)^2\frac{\sigma_u^2}{u_H^2}}}$$

multiply by $\frac{u_H}{u_H}$

$$\sigma_v = \tan\left(\frac{\pi}{2K}\right)\frac{\left(u_H^2 - (2\ln 2)\sigma_u^2\right)}{\sqrt{(2\ln 2)\,u_H^2 - (2\ln 2)^2\sigma_u^2}}$$

$$= \frac{\tan(\pi/2K)}{\sqrt{2\ln 2}}\;\frac{u_H^2 - (2\ln 2)\,\sigma_u^2}{\sqrt{u_H^2 - (2\ln 2)\sigma_u^2}}$$

$$\Rightarrow \sigma_v = \frac{\tan(\pi/2K)}{\sqrt{2\ln 2}}\sqrt{u_H^2 - (2\ln 2)\sigma_u^2}$$

$$\Rightarrow \boxed{\sigma_v = \tan(\pi/2K)\sqrt{\frac{u_H^2}{2\ln 2} - \sigma_u^2}} \qquad \#$$

Which is the same as eqn (3).