# STATISTICAL THRESHOLD DESIGN FOR THE TWO-STATE SIGNAL-DEPENDENT RANK ORDER MEAN FILTER

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## ABSTRACT

The signal-dependent rank order mean (SD-ROM) £lter is effective at removing high levels of impulse noise from 2-D scalar-valued signals. Excellent results have been presented for both a two-state and a multi-state version of the £lter. However, implementation of the two-state SD-ROM £lter requires the selection of a set of threshold values. In this paper, we propose a method for choosing the thresholds based on a statistical characterization of an input image. The method approximates the histogram of an image with a weighted sum of Gaussian distributions. Using the statistical model and the input distributions, the likelihood of correctly identifying impulses is estimated as a function of the thresholds. By maximizing the likelihood of correct detection, optimal thresholds are predicted. The performance of the algorithm using the predicted thresholds is compared to the optimal performance found using a brute-force search.

## 1. INTRODUCTION

One of the most common image processing tasks involves the removal of noise from images. Noise can be introduced during image capture, during transmission, or during storage. For design purposes, noise sources are frequently approximated by random variables with a known probability distribution. As a result, many different types of £lters have been developed to handle different kinds of noise sources [1, 2]. One common noise model corrupts a signal by introducing impulses. In this case, most of the original samples are unaltered, but the few corrupted samples may vary drastically. One group of £lters, called decision-based £lters [2, 3] or state-conditioned £lters [4], estimates the state of the sample in question. If the sample is determined to be uncorrupted, it is passed through the £lter unchanged. If the sample is corrupted, an appropriate estimate is chosen to replace it.

The signal dependent rank order mean (SD-ROM) £lter has been shown to be effective at removing impulses from 2-D scalar-valued signals [4]. Excellent results were shown for both a two-state and a multi-state version of the £lter. Although a speci£c design method was developed for the multi-state SD-ROM, no method was proposed in [4] for £nding the thresholds required by the simpler two-state algorithm.

In this paper, we present a probabilistic model for predicting the detection performance of the SD-ROM £lter as a function of the thresholds, image distribution, and noise model. The image distribution is modeled using a Gaussian decomposition of its histogram. A uniformly distributed noise model is assumed for the impulse noise. By maximizing the detection performance with respect to the thresholds, we predict a set of optimal thresholds for an image and level of corruption. To test the predictions, the thresholds are compared to thresholds found by a brute-force search on simulated corrupted images.

## 2. BACKGROUND

### 2.1. SD-ROM Filter

The SD-ROM algorithm works in two steps, detection and estimation. First, a small neighborhood around a sample is used to determine if the central sample is corrupted. If the detection step decides the central sample is corrupted, a new value is estimated for the central sample using the samples in the surrounding window. If the detection step decides the central sample is uncorrupted, the sample is passed unchanged.

Although the surrounding window can be of arbitrary size and shape, in practice a three by three window is used in most applications. The center pixel within the window is designated $x(\mathbf{n})$. The surrounding eight values are labeled $x_1(\mathbf{n}) \ldots x_8(\mathbf{n})$ where $x_1(\mathbf{n})$ is the upper left value in the

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window and $x_8(\mathbf{n})$ is the lower right value. The remaining values are labeled by scanning across rows in the window, skipping the center value. The algorithm can then be summarized as follows:

1. Sort pixel values in the surrounding window from smallest to biggest:

$$x_1(\mathbf{n}), \ldots, x_8(\mathbf{n}) \rightarrow r_1(\mathbf{n}) \leq \ldots \leq r_8(\mathbf{n}) \quad (1)$$

2. Compute the rank-order differences, $i = 1, \ldots, 8$:

$$d_i(\mathbf{n}) = \begin{cases} r_i(\mathbf{n}) - x(\mathbf{n}), & i = 1, \ldots, 4 \\ x(\mathbf{n}) - r_i(\mathbf{n}), & i = 5, \ldots, 8, \end{cases} \quad (2)$$

3. Threshold and replace, if necessary, $j = 1, \ldots, 4$:

$$y(\mathbf{n}) \equiv \begin{cases} m(\mathbf{n}), & d_j(\mathbf{n}) > T_j, \\ m(\mathbf{n}), & d_{9-j}(\mathbf{n}) > T_j, \\ x(\mathbf{n}), & \text{otherwise.} \end{cases} \quad (3)$$

The ordered window values are $r_1(\mathbf{n}), \ldots, r_8(\mathbf{n})$. The *rank ordered differences* are $d_1(\mathbf{n}), \ldots, d_8(\mathbf{n})$. $m(\mathbf{n})$ is the *rank order mean* and is the average of $r_4(\mathbf{n})$ and $r_5(\mathbf{n})$. This value replaces the center pixel if any of the thresholds $T_1, \ldots, T_4$ are exceeded. Because the difference calculation results in a signed difference and not an absolute difference, the thresholds are restricted to values greater than or equal to zero. The signed difference also gives the £lter a step response similar to the median £lter.

### 2.2. Noise Model

Although the SD-ROM £lter can effectively remove many types of corruption, in this paper we only consider a particular kind of impulse noise. Impulse noise is de£ned as

$$x(\mathbf{n}) = \begin{cases} v(\mathbf{n}), & \text{with probability } 1-p \\ \eta(\mathbf{n}), & \text{with probability } p, \end{cases} \quad (4)$$

where $v(\mathbf{n})$ is the original image value, $\eta(\mathbf{n})$ is a sample of an identically distributed, independent random process with a uniform probability density function, and $p$ is the probability of corruption.

## 3. THRESHOLD OPTIMIZATION

### 3.1. Statistical Characterization

To create a statistical characterization of the SD-ROM algorithm, we assume that each sample window contains nine random variables. Each sample is an instance of either the background (original image) distribution or the impulse noise distribution. We also assume that the proportion of each distribution (percent corruption) is also known.

Given these assumptions, we derive the probabilities that an impulse will be detected given that the center is corrupted (desired) and that an impulse will be detected given that the center is uncorrupted (undesired).

Preliminary work in this direction was presented in [5]. The up-to-date equations, which are rather lengthy, are included in Table 1. The derivation was done in the following stages:

1. The probability density function (pdf) for the sorted values in the surrounding window is derived, given the rank $n$, the number of impulses in the window $N_c$, the background distribution, and the noise distribution.
2. The probability mass functions (pmf) for the difference between each sorted window value and the center value is then found, given the rank, the number of impulses, and whether or not the center value is an impulse.
3. For each threshold, the probability that the difference will not exceed the threshold is found, given the state of the center pixel (corrupted or uncorrupted). This step requires a summation over all possible values of $N_c$ and two different ranks $n$ for each threshold.
4. The probabilities of passing the center with the individual thresholds are combined into the probability of passing the center value for a set of thresholds, given the state of the center value.
5. The results from each center value state are weighted by the probability of each state and combined. The detection performance is a sum of the probabilities that an impulses will be found and that non-impulses will not be found.

Using the statistical model, the uniform impulse noise model, and an assumed image model, we arrive at the thresholds with the greatest probability of correct detection by numerically maximizing with respect to the thresholds.

### 3.2. Background Distribution

At £rst glance, using a statistical model for a spatial £lter may not seem like a good idea. The assumption that each pixel value represents an independent random value with a background distribution ignores the fact that image values are normally spatially correlated. However, the SD-ROM algorithm sorts the values in its window. Therefore, all spatial information within each window is lost.

One approach to estimating the background distribution required by the statistical model would be to use the image histogram. This method does not work very well. The histograms tend to have wide distributions, which lead to large thresholds. Using the histogram ignores the fact that the window sweeps across the image, moving from region to region, where each region has a different background distribution. The background distribution may be narrow within each region, but vary signi£cantly between regions, giving the overall wide distribution of the histogram. Narrower distributions lead to smaller thresholds. Therefore, a weighted combination of narrow distributions should result in a better background distribution.

The best approach would be to segment the image into regions, £nd the background distributions for each region, and estimate the proportion of the image with each distribution. Then the detection probability curves for each region could be calculated, weighted by its proportion, and combined into an overall detection probability curve. Maximization of this curve would provide a better estimate of the optimal thresholds.

As a £rst step towards this goal, we implemented the histogram decomposition technique described in [6], with some minor modi£cations. In this method, each image is assumed to be composed of regions with roughly Gaussian distributions of different means and variances. The image histogram is £tted with a set of Gaussian curves. An example for the Lena image is shown in Figure 1. This method still does not use any actual spatial or regional information from the image. It does not work well for regions with bimodal distributions.

## 4. EXPERIMENTAL RESULTS

We used the set of Gaussian curves estimated from the Lena image histogram as inputs to the statistical model. The resulting distributions were weighted, added, and maximized to £nd the thresholds with the best detection performance. The procedure was applied for corruption percentages of 5, 10, 15, 20, 25, 30, 40, and 50 percent. To see if the process actually represents an improvement over using the histogram, the optimum thresholds were also found using the histogram as the background distribution.

Several versions of the Lena image with impulse noise at the various corruption percentages were also created. The simulated images were used to £nd the optimum thresholds by brute-force and to measure the actual performance of the thresholds predicted using the statistical model.

Figure 2 contains plots of the results. Part (a) contains plots of all four thresholds versus the corruption percentages for each of the three approaches. Although the Gaussian approach does not predict the optimal thresholds exactly, it does come much closer than the histogram approach. Part (b) plots the realized performance for each set of thresholds on the corrupted images versus the corruption percentages. Once again, the Gaussian approach does not achieve the optimum, but comes much closer than the histogram results.

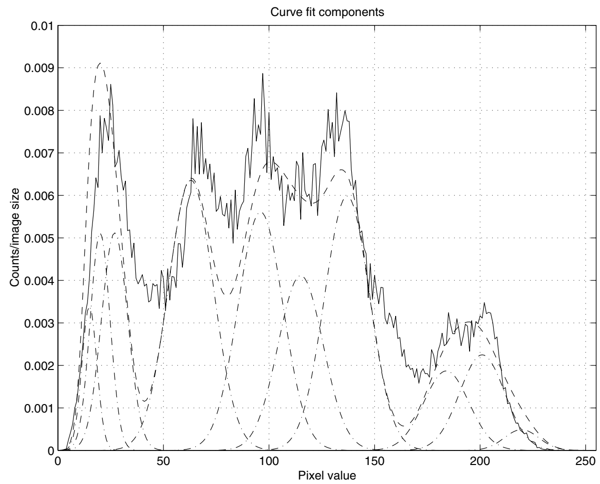


**Fig. 1**. Histogram decomposition. The original Lena histogram (solid), Gaussian approximation (dashed), and component curves (dash-dot) are shown.

## 5. REFERENCES

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**Table 1**. SD-ROM Statistical Model Equations.

1. Sorted value probability for two independent distributions (corruption and background image):

$$f_{(n)}(x|N_c) = \sum_{i=1}^{N} \sum_{j=\alpha}^{A} \sum_{k=\beta}^{B} \sum_{l=\gamma}^{\Gamma} \binom{N_c}{j\,,l} \binom{N_b}{(i-j)\,,(k-l)} \mathrm{P}_{ce}^{j}\mathrm{P}_{be}^{(i-j)}\mathrm{P}_{cl}^{l}\mathrm{P}_{bl}^{(k-l)}\mathrm{P}_{cg}^{(N_c-j-l)}\mathrm{P}_{bg}^{(N_b-(i-j)-(k-l))}$$

where,

$$\begin{array}{lllllllll} \alpha & = & \max(0, i-N_b), & A & = & \min(i, N_c), & \mathrm{P}_{ce}, \mathrm{P}_{cl}, \mathrm{P}_{cg} & = & \text{Prob(impulse)} = x, < x, \text{or} > x, \\ \beta & = & \max(0, n-i), & B & = & \min(n-1, N-i), & \mathrm{P}_{be}, \mathrm{P}_{bl}, \mathrm{P}_{bg} & = & \text{Prob(background)} = x, < x, \text{or} > x, \\ \gamma & = & \max(0, k-N_b+i-j), & \Gamma & = & \min(k, N_c-j), & n & = & \text{rank}, \\ N & = & \text{window size}, & N_c & = & \text{impulse number, and} & N_b & = & N - N_c. \end{array}$$

2. Probability mass functions (PMFs) for the rank-ordered differences:

$$F_{c,(n)}(z|N_c) \quad = \quad \sum_{x=0}^{255} F_c(x+z) f_{(n)}(x|N_c) \quad \text{and} \quad F_{u,(n)}(z|N_i) \quad = \quad \sum_{x=0}^{255} F_b(x+z) f_{(n)}(x|N_c)$$

where,

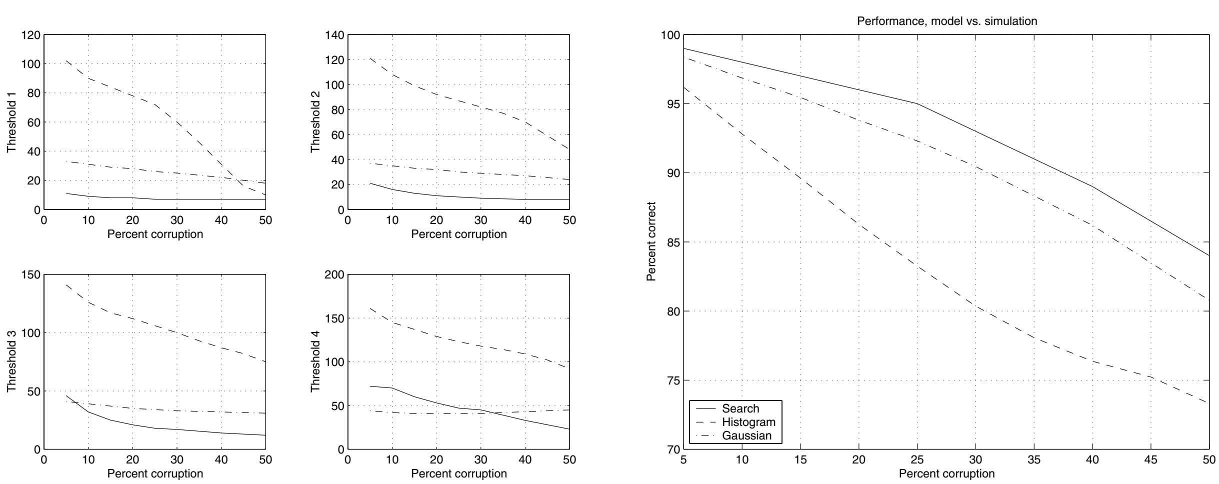
$$F_c(x) = \text{the noise PMF and } F_b(x) = \text{the background image PMF.}$$

3. Probability of center passing a single threshold:

$$\begin{aligned} \text{Prob}_{\text{pass}}(T_i|\text{corrupted}) &= \sum_{j=0}^{8} P_c(j)[F_{c,(9-i)}(T_i|j) - F_{c,i}((-T_i-1)|j)] \\ \text{Prob}_{\text{pass}}(T_i|\text{uncorrupted}) &= \sum_{j=0}^{8} P_c(j)[F_{u,(9-i)}(T_i|j) - F_{u,i}((-T_i-1)|j)] \end{aligned}$$

4. Correct detection probability:

$$\text{Prob}_{\text{correct}}(\mathbf{T}) = p * (1 - \text{Prob}_{\text{all pass}}(\mathbf{T}|\text{corrupt})) + (1-p) * \text{Prob}_{\text{all pass}}(\mathbf{T}|\text{uncorrupt})$$




(a) (b)

**Fig. 2**. Performance comparison. The optimal thresholds found by direct search (solid), the statistical model using the Lena histogram (dashed), and the statistical model using the Gaussian decomposition (dash-dot) are shown in (a). The performance achieved by each set of thresholds in simulated tests are shown in (b).